The extended coset leader weight enumerator

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Abstract

This paper is a report on the ongoing research concerning the extended coset leader weight enumerator using the theory of arrangements of hyperplanes, geometric lattices and characteristic polynomials.

1 Introduction

The probability of error in error-detection can be expressed in terms of the weight enumerator of a code [11], and for error-correction the coset leader weight enumerator is used [12]. The coset leader weight enumerator is also used in steganography to compute the average of changed symbols [13]. The computation of the weight enumerator of a code is NP-hard [2, 18]. The complexity of computing the coset leader weight enumerator of a code is considered extremely difficult [5]. The size of lists of nearest codewords is considered in the list decoding of Reed-Solomon codes [8, 15]. This motivates the definition of the list weight enumerator and its extension.

Let G be a generator matrix of the code C of length n over $\mathbb{F}_q$. Let $C \otimes \mathbb{F}_{q^m}$ in $\mathbb{F}_{q^m}^n$ be the $\mathbb{F}_{q^m}$-linear code with G as generator matrix, where $\mathbb{F}_{q^m}$ is the extension of $\mathbb{F}_q$ of degree m. This motivates to consider q^m as a variable in the following definition. See [6, 7]. The *extended weight enumerator* $W_C(X, Y, T)$ of a linear code of length n and is a homogeneous polynomial in X and Y of degree n with coefficients $A_w(T)$ that are integral polynomials in T:

$$W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w,$$

such that $A_w(q^m)$ is the number of codewords of weight w in $C \otimes \mathbb{F}_{q^m}$.

2 Codes, projective systems and arrangements

Let q be a power of a prime. Denote the finite field with q elements by $\mathbb{F}_q$. A *projective system* $\mathcal{P} = (P_1, \ldots, P_n)$ in $\mathbb{P}^r(\mathbb{F}_q)$, the projective space over $\mathbb{F}_q$ of dimension r is an enumeration of points P_j in this projective space, such that not all these points lie in a hyperplane. See [16, §1.1.2] Let P_j be given by be homogeneous coordinates $(p_{0j} : p_{1j} : \cdots : p_{rj})$. Let $G_{\mathcal{P}}$ be the $(r+1) \times n$ matrix with $(p_{0j}, p_{1j}, \ldots, p_{rj})^T$ as j-th column. If $\mathbb{F}_q$ is a finite field, then $G_{\mathcal{P}}$ is the generator matrix of a nondegenerate code over $\mathbb{F}_q$ of length n and dimension $r+1$, since not all points lie in a hyperplane. Conversely, let G be a generator matrix of a nondegenerate code C of dimension k over $\mathbb{F}_q$. So G has no zero columns. Take the columns of G as homogeneous coordinates of points in $\mathbb{P}^{k-1}(\mathbb{F}_q)$. This gives the projective system $\mathcal{P}_G$ over $\mathbb{F}_q$ of G. See [9, 16, 17].

Proposition 2.1 *Let C be a nondegenerate code over $\mathbb{F}_q$ of length n and dimension k with generator matrix G. Let $\mathcal{P}_G$ be the projective system of G. The code has minimum distance d if and only if $n-d$ is the maximal number of points of $\mathcal{P}_G$ in a hyperplane of $\mathbb{P}^{k-1}(\mathbb{F}_q)$.*

An n-tuple $(H_1, \ldots, H_n)$ of hyperplanes in $\mathbb{F}_q^k$ is called an *arrangement* in $\mathbb{F}_q^k$. The arrangement is called *central* if all the hyperplanes contain $\{0\}$. A central arrangement is called *essential* if the intersection of all its hyperplanes is equal to $\{0\}$. In case of an essential arrangement one considers the hyperplanes in $\mathbb{P}^{k-1}(\mathbb{F}_q)$. Note that projective systems and arrangements are dual notions and that there is a one-to-one correspondence between generalized equivalence classes of non-degenerate $[n,k,d]$ codes over $\mathbb{F}_q$, equivalence classes of projective systems over $\mathbb{F}_q$ of n points in $\mathbb{P}^{k-1}(\mathbb{F}_q)$ and equivalence classes of essential arrangements of n hyperplanes in $\mathbb{P}^{k-1}(\mathbb{F}_q)$.

3 The characteristic polynomial of an arrangement

The translation for an arrangement of Proposition 2.1 gives:

Proposition 3.1 *Let C be a nondegenerate code over $\mathbb{F}_q$ with generator matrix G. Let $\mathbf{c}$ be a codeword $\mathbf{c} = \mathbf{x}G$ for some $\mathbf{x} \in \mathbb{F}_q^k$. Then $n - \mathrm{wt}(\mathbf{c})$ is equal to the number of hyperplanes in $\mathcal{A}_G$ through $\mathbf{x}$.*

The number A_w of codewords of weight w equals the number of points that are on exactly $n-w$ of the hyperplanes in $\mathcal{A}_G$, by Proposition 3.1. In particular A_n is equal to the number of points that is in the complement of the union of these hyperplanes in $\mathbb{F}_q^k$. This number can be computed by the *principle of inclusion/exclusion*:

$$A_n = q^k - |H_1 \cup \cdots \cup H_n| =$$

$$q^k + \sum_{w=1}^{n} (-1)^w \sum_{i_1 < \cdots < i_w} |H_{i_1} \cap \cdots \cap H_{i_w}|.$$

This counting principle is formalized in the notion of a characteristic polynomial of a geometric lattice. See [4, 14].

Let $\mathcal{A} = (H_1, \ldots, H_n)$ be an essential arrangement over $\mathbb{F}_q$ of hyperplanes in the vectorspace $V = \mathbb{F}_q^k$. Let $L = L(\mathcal{A})$ be the collection of all nonempty intersections of elements of $\mathcal{A}$. By definition V is the empty intersection. Define the partial order $\leq$ by the reverse inclusion:

$$x \leq y \text{ if and only if } y \subseteq x.$$

Then V is the smallest element which is denoted by 0, and the zero subspace is the largest element denoted by 1. Let r be a nonnegative integer. A *chain of length r from x to y* is a sequence of elements $x_0, x_1, \cdots, x_r$ in L such that $x = x_0 < x_1 < \ldots < x_r = y$. Let $c_r(x,y)$ denote the number of chains of length r from x to y. Now $c_r(x,y)$ is finite, since L is finite. The *Möbius function* of L, denoted by μ_L or μ is defined by

$$\mu(x,y) = \sum_{r=0}^{\infty} (-1)^r c_r(x,y).$$

and satisfies the following recurrence relations:

(M.1) $\mu(x,x) = 1$.

(M.2) If $x < y$, then $\sum_{x \le z \le y} \mu(x,z) = \sum_{x \le z \le y} \mu(z,y) = 0$.

Define

$$x \vee y := x \cap y \quad \text{and} \quad x \wedge y := \cap\{z | x \cup y \subseteq z\}.$$

Then L is a *lattice*, since L has a smallest element 0 and a largest element 1, and $x \vee y$ is the smallest upper bound and $x \wedge y$ the largest lower bound in L for all x, y in L.
Let $r(x)$ be the codimension of x in V. So $r(V) = 0$, and $r(x) = 1$ if and only if $x = H_j$ for some j. Hence the *atoms* of $L(\mathcal{A})$ are $H_1, \ldots, H_n$. The *rank* $r(L)$ is equal to $r(1) = k$, the dimension of V.
Now $L(\mathcal{A})$ is a geometric lattice with rank function r, since the following conditions hold:

(GL.1) For every x in L, $x \neq 0$ there exist atoms $a_1, \ldots, a_r$ such that $x = a_1 \vee \cdots \vee a_r$, and the smallest possible r is called the *rank* of x and is denoted by $r_L(x)$ or $r(x)$.

(GL.2) If $x < y$, then $r(x) < r(y)$ for all $x, y \in L$.

(GL.3) $r(x \wedge y) + r(x \vee y) \le r(x) + r(y)$ for all $x, y \in L$.

Let Let $L_j = \{x \in L | r(x) = j\}$. Then L_j is called the *level* of L. The *Hasse diagram* of L is a graph that has the elements of L as vertices. If $x, y \in L$, $x < y$ and $r(y) = r(x) + 1$ then x and y are connected by an edge. So only elements between two consecutive levels L_j and L_{j+1} are connected by an edge.
Define $L_x = \{y \in L | x \le y\}$ and $L^x = \{y \in L | y \le x\}$.

The *characteristic polynomial* $\chi_L(T)$ and the *Poincaré polynomial* π_L of L are defined by:

$$\chi_L(T) = \sum_{x \in L} \mu_L(x) T^{r(L) - r(x)}, \quad \text{and} \quad \pi_L(T) = \sum_{x \in L} \mu_L(x) (-T)^{r(x)}.$$

So $\mu(L) = \chi_L(0)$, and $\chi_L(1) = 0$ if and only if L consists of one element $0 = 1$.
Furthermore $\chi_L(T) = T^{r(L)} \pi_L(-T^{-1})$. For the following we refer to [14, Theorem 2.69].

Proposition 3.2 *Let q be a prime power, and let $\mathcal{A} = (H_1, \ldots, H_n)$ be an arrangement in $\mathbb{F}_q^k$ and $L = L(\mathcal{A})$ its associated geometric lattice. Then*

$$\chi_{\mathcal{A}}(q^m) = |\mathbb{F}_{q^m}^k \setminus (H_1 \cup \ldots \cup H_n)|.$$

A nondegenerate code C over $\mathbb{F}_q$ in $\mathbb{F}_q^n$ with generator matrix G gives rise to the arrangement $\mathcal{A}_G$, and its characteristic polynomial will be denoted by χ_C, since it does not depend on the chosen generator matrix G of C.

Proposition 3.3 *Let C be a nondegenerate $\mathbb{F}_q$-linear code. Then*

$$A_n(T) = \chi_C(T).$$

Consider an essential arrangement $\mathcal{A} = (H_1, \ldots, H_n)$ over $\mathbb{F}_q$ in $\mathbb{F}_q^k$. Then $\chi_{\mathcal{A}}(q^m)$ counts the number of elements in the complement of $\cup_{j=1}^n H_j$ in $\mathbb{F}_{q^m}^k$. Consider the *stratification* of the affine space $\mathbb{A}^k$ of dimension k by:

$$\mathcal{X}_{-1} \subset \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_{k-1} \subset \mathcal{X}_k,$$

where $\mathcal{X}_k = \mathbb{A}^k$, $\mathcal{X}_{k-1} = H_{\mathcal{A}} = \cup_{j=1}^n H_j$, $\mathcal{X}_0 = \{0\}$ and $\mathcal{X}_{-1} = \emptyset$, and more generally

$$\mathcal{X}_{k-t} = \bigcup_{r(\cap_{i=1}^t H_{j_i})=t} (H_{j_i} \cap \cdots \cap H_{j_t}) .$$

Then $\mathcal{X}_j$ is a closed affine subvariety of $\mathbb{A}^k$ of dimension j. For a variety $\mathcal{X}$ that is defined over $\mathbb{F}_q$, we denote by $\mathcal{X}(\mathbb{F}_q)$ the set of points in $\mathcal{X}$ that have coordinates in $\mathbb{F}_q$.

Proposition 3.4 *Let $\mathcal{A}$ be an essential arrangement. Let $L = L(\mathcal{A})$ be the geometric lattice of $\mathcal{A}$. Let $\chi_{L,j}(T) = \sum_{x \in L, r(x)=j} \chi_{L_x}(T)$. Then*

$$\chi_{L,j}(q^m) = |(\mathcal{X}_j \setminus \mathcal{X}_{j-1})(\mathbb{F}_{q^m})|.$$

4 Coset leader and list weight enumerator

Let C be a linear code of length n over $\mathbb{F}_q$. Let $\mathbf{y} \in \mathbb{F}_q^n$. The weight of the coset $\mathbf{y} + C$ is defined by

$$\mathrm{wt}(\mathbf{y} + C) = \min\{ \mathrm{wt}(\mathbf{y} + \mathbf{c}) \mid \mathbf{c} \in C \}.$$

A *coset leader* is a choice of an element $\mathbf{y} \in \mathbb{F}_q^n$ of minimal weight in its coset, that is $\mathrm{wt}(\mathbf{y}) = \mathrm{wt}(\mathbf{y} + C)$. Let α_i be the number of cosets of C that are of weight i. Let λ_i be the number of $\mathbf{y}$ in $\mathbb{F}_q^n$ that are of minimal weight i in its coset. Then $\alpha_C(X, Y)$, the *coset leader weight enumerator* of C and $\lambda_C(X, Y)$, the *list weight enumerator* of C are polynomials defined by

$$\alpha_C(X, Y) = \sum_{i=0}^n \alpha_i X^{n-i} Y^i \quad \text{and} \quad \lambda_C(X, Y) = \sum_{i=0}^n \lambda_i X^{n-i} Y^i$$

See [5, 12]. The *covering radius* $\rho(C)$ of C is the maximal i such that $\alpha_i(C) \neq 0$.

We have $\alpha_i = \lambda_i = \binom{n}{i}(q-1)^i$ for all $i \leq (d-1)/2$, where d is the minimum distance of C. The coset leader weight enumerator gives a formula for the *probability of error*, that is the probability that the output of the decoder is the wrong codeword. In this decoding scheme the decoder uses the chosen coset leader as the error vector. See [12, Chap.1 §5]. The list weight enumerator is of interest in case the decoder has as output the list of all nearest codewords [8, 15].

Consider the functions $\alpha_i(T)$ and $\lambda_i(T)$ such that $\alpha_i(q^m)$ and $\lambda_i(q^m)$ are equal to the number of cosets of weight i and the number of elements in $\mathbb{F}_{q^m}^n$ of minimal weight i in its coset, respectively with respect to the extended coded $C \otimes \mathbb{F}_{q^m}$. Define the *extended coset leader weight enumerator* and the *extended list weight enumerator*, respectively by:

$$\alpha_C(X, Y, T) = \sum_{i=0}^n \alpha_i(T) X^{n-i} Y^i \quad \text{and} \quad \lambda_C(X, Y, T) = \sum_{i=0}^n \lambda_i(T) X^{n-i} Y^i$$

In [5, Theorem 2.1] it is shown that the function $\alpha_i(T)$ is determined by finitely many data for all extensions of $\mathbb{F}_q$. This shows by Lagrange interpolation, that the $\alpha_i(T)$ are

polynomials in the variable T. In fact, let C be an $[n,k]$ code over $\mathbb{F}_q$. Then there are well defined nonnegative integers F_{ij} such that

$$\alpha_C(X,Y,T) = 1 + \sum_{i=1}^{n-k}\sum_{j=1}^{n-k} F_{ij}(T-1)(T-q)\cdots(T-q^{j-1})X^{n-i}Y^i$$

This is similar to the following expression of the extended weight enumerator in terms of the generalized weight enumerator. See [5, 6, 10].

$$A_w(T) = \sum_{r=0}^{k}\sum_{j=1}^{r}(T-1)(T-q)\cdots(T-q^{j-1})A_w^r.$$

Although the extended weight enumerator, the Tutte polynomial and the matroid of a code contain a lot of information of a code, they do not determine the coset leader weight enumerator or even the covering radius of a code. See [3]. For instance all $[n,k,n-k+1]$ codes over $\mathbb{F}_q$ are MDS and have the same generalized weight enumerator, uniform matroid and Tutte polynomial but the covering radius varies for fixed n,k and q.

There is a one-to-one correspondence between cosets and syndromes. It is a well known fact that a coset leader corresponds to a minimal way to write its syndrome as a linear combination of the columns of a parity check matrix. This idea is formalized as follows. Let H be a parity check matrix of a $[n,k]$ code C over $\mathbb{F}_q$. Let $\mathbf{y}$ be a received word. Then $\mathbf{s} = H\mathbf{y}^T$ is the *syndrome* of this word with respect to H. Define the *weight of* $\mathbf{s}$ *with respect to* H also called the *syndrome weight* of $\mathbf{s}$, by $\mathrm{wt}_H(\mathbf{s}) = \mathrm{wt}(\mathbf{y}+C)$. Then α_i is the number of syndromes in $\mathbb{F}_q^{n-k}$ with respect to H that are of weight i. See [5, Definition 2.1].

Let $\mathbf{h}_j$ be the j-th column of H. Let $J \subseteq \{1,\ldots,n\}$. Let V_J be the vector subspace of $\mathbb{F}_q^{n-k}$ that is generated by the vectors $\mathbf{h}_j^T$, $j \in J$. Let

$$\mathcal{V}_t = \bigcup_{|J|=t} V_J.$$

Proposition 4.1 *Let* $\mathbf{s}$ *in* $\mathbb{F}_q^{n-k}$ *be a syndrome with respect to* H*. Then*

$$\mathrm{wt}_H(\mathbf{s}) = t \quad \textit{if and only if} \quad \mathbf{s} \in \mathcal{V}_t \setminus \mathcal{V}_{t-1}.$$

Let J consist of t elements. If V_J has dimension t', then there is a $J' \subseteq J$ consisting of t' elements such that $\mathbf{h}_i$, $i \in J'$ are independent. So $V_J = V_{J'}$. Now V_J is a subspace of the column space of H, which has dimension $n-k$. Hence there is an $I \subseteq [n]$ consisting of $n-k$ elements such that and $J' \subseteq I$ and $\mathbf{h}_i$, $i \in I$ are independent. So

$$V_J = \bigcap_{i \in (I \setminus J')} V_{I\setminus\{i\}}.$$

is an intersection of the $n-k-t'$ hyperplanes $V_{I\setminus\{i\}}$.

5 The derived code

Let H be a parity check matrix of a $[n,k]$ code C over $\mathbb{F}_q$. Consider the arrangement of all hyperplanes V_I where I consists of $n-k-1$ elements such that the $\mathbf{h}_i$, $i \in I$ are independent. Let N be the number of such hyperplanes, without multiplicities. This arrangement is essential and gives the *(reduced) derived code* code $D(C)$ of dimension $n-k$ with generator matrix $D(G)$ of size $(n-k) \times N$ whose columns correspond to the hyperplanes of the arrangement and where the entries of a column correspond to the coefficients of the defining equation of the hyperplane. The matrix is reduced in the sense that no columns is a scalar multiple of another column, that is all the N hyperplanes of the arrangement are mutually distinct. This gives rise to the stratification of the affine space $\mathbb{A}^{n-k}$ of dimension $n-k$ as explained in Section 3:

$$\mathcal{X}_{-1} \subset \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_{n-k-1} \subset \mathcal{X}_{n-k},$$

Furthermore we have the stratification

$$\mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{n-k-1} \subset \mathcal{V}_{n-k},$$

of Section 4. Now $\mathcal{V}_j \subseteq \mathcal{X}_j$ for all j. That is to say: all components of $\mathcal{V}_t$ are intersections of $n-k-t$ hyperplanes of the arrangement, but it is not always the case that all such intersections of dimension t are components of $\mathcal{V}_t$. This can be formalized for $\alpha_j(T)$ by the following analogy of Proposition 3.4.

Proposition 5.1 *Let $\mathcal{A}$ be the arrangement of $D(C)$ and let $L = L(\mathcal{A})$ be the geometric lattice of $\mathcal{A}$. Let x_j be the linear subspace generated by $\mathbf{h}_j$ and define $M = \{x \in L \mid x = x_{j_1} \wedge \cdots \wedge x_{j_t}\}$. Let $r^*(x) = \max\{r(y) \mid y \in M, y \leq x\}$. Then*

$$\alpha_j(T) = \sum_{x \in L, r^*(x)=j} \chi_{L_x}(T).$$

Note that x is the subspace of V generated by $x_{j_1}, \ldots, x_{j_t}$, and that $r^*(x) = r(x)$ if and only if $x \in M$. A similar expression can be given for $\lambda_i(T)$.

We have $\alpha_i(T) = \lambda_i(T) = \binom{n}{i}(T-1)^i$ for all $i \leq (d-1)/2$, where d is the minimum distance of C. The polynomials $A_i(T)$, $\alpha_i(T)$ and $\lambda_i(T)$ are divisible by $T-1$ for all $i > 0$. Let $i(C)$ be the number of *information sets* of C. Then $\lambda_{n-k}(T) = i(C)\alpha_{n-k}(T)$.

Several examples are considered:

Let $C = \mathbb{F}_q^n$. Then $\lambda_C(X,Y,T) = \alpha_C(X,Y,T) = X^n$.

Let $C = \{0\}$. Then $\lambda_i(T) = \alpha_i(T) = \binom{n}{i}(T-1)^i X^{n-i}Y^i$ and
$\lambda_C(X,Y,T) = \alpha_C(X,Y,T) = (X + (T-1)Y)^n$.

Let C be the dual of the $[n,1,n]$ repetition code. Then
$\lambda_C(X,Y,T) = X^n + n(T-1)X^{n-1}Y$ and $\alpha_C(X,Y,T) = X^n + (T-1)X^{n-1}Y$.

Let C be the $[n,1,n]$ repetition code. Then this code has not such an easy description of $\lambda_C(X,Y,T)$ and $\alpha_C(X,Y,T)$ as the previous example. Apart from the known expressions for $\lambda_i(T)$ and $\alpha_i(T)$ for $i \leq (n-1)/2$ that hold for every code we have that $\lambda_{n-1}(T) = n\alpha_{n-1}(T)$ and $\alpha_{n-1}(T) = (T-1)(T-2)\cdots(T-n+1)$.

Let C be the binary Hamming code of length 7. Then $\lambda_i(T) = \alpha_i(T)$ for $i \leq 1$, and $\alpha_0(T) = 1$, $\alpha_1(T) = 7(T-1)$, $\lambda_2(T) = 3\alpha_2(T) = 21(T-1)(T-2)$ and
$\lambda_3(T) = 28\alpha_3(T) = 28(T-1)(T-2)(T-4)$.
So $\rho(C) = 1$, $\rho(C \otimes \mathbb{F}_4) = 2$ and $\rho(C \otimes \mathbb{F}_{2^m}) = 3$ for $m \geq 3$.

6 MacWilliams type property for duality

Research Problem 5.1 in [12, Chapter 5] asked whether the coset leader weight enumerator of C determines the coset leader weight enumerator of $C^\perp$, as is the case for the ordinary weight enumerator by the MacWilliams relations. This problem has a negative answer by [1]. The authors give three binary [15,3,7] codes that have the same coset leader weight enumerator, but the dual codes have mutually distinct coset leader weight enumerators. In fact a much smaller counterexample will do.

The two codes of length 3 with parity check matrices $H_1 = (110)$ and $H_2 = (111)$ both have the same extended coset leader weight enumerator $X^3 + (T-1)X^2Y$. But their dual codes have distinct extended coset leader weight enumerator, since

$$\alpha_{C_1^\perp}(X,Y,T) = X^3 + 2(T-1)X^2Y + (T-1)XY^2$$

$$\alpha_{C_2^\perp}(X,Y,T) = X^3 + 3(T-1)X^2Y + (T-1)(T-2)XY^2.$$

Remark that the code $C_1^\perp$ is degenerate. A non degenerate counterexample is obtained as follows. Let C_3 and C_4 be the two $[6,3]$ codes over $\mathbb{F}_2$ with generator matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The next table shows the coefficients of the extended coset leader weight enumerator and the extended list weight enumerator of the codes and their duals. The values for $i = 0$ are left out, since they are all equal to 1 because of the zero word.

	i	C_3	C_4
$\alpha_{C,i}$	1	$5(T-1)$	$5(T-1)$
	2	$2(T-1)(3T-5)$	$2(T-1)(3T-5)$
	3	$(T-1)(T-2)(T-3)$	$(T-1)(T-2)(T-3)$
$\alpha_{C^\perp,i}$	1	$4(T-1)$	$5(T-1)$
	2	$3(T-1)(2T-3)$	$2(T-1)(3T-5)$
	3	$(T-1)(T-2)(T-3)$	$(T-1)(T-2)(T-3)$
$\lambda_{C,i}$	1	$6(T-1)$	$6(T-1)$
	2	$2(T-1)(7T-12)$	$2(T-1)(7T-11)$
	3	$12(T-1)(T-2)(T-3)$	$13(T-1)(T-2)(T-3)$
$\lambda_{C^\perp,i}$	1	$6(T-1)$	$6(T-1)$
	2	$13(T-1)^2$	$2(T-1)(7T-11)$
	3	$12(T-1)(T-2)(T-3)$	$13(T-1)(T-2)(T-3)$

We see that the extended coset leader weight enumerator of the two codes are equal, but none of the other polynomials, so they are not defined by the extended coset leader weight enumerator.

Question: Is the extended list weight enumerator of C determined by the corresponding enumerator of its dual?

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