Generalized weight enumerators

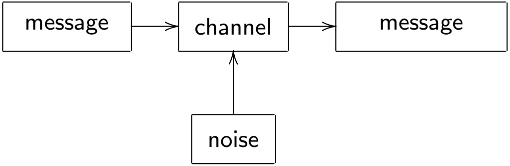
Relinde Jurrius

Technische Universiteit Eindhoven

November 26, 2008

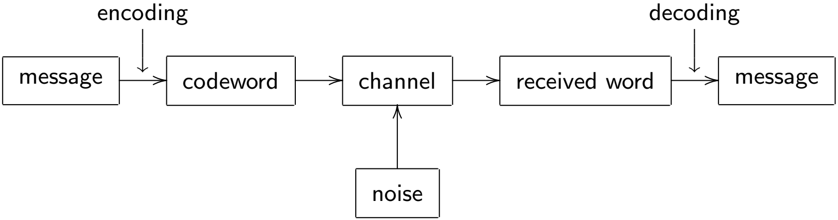
Outline

What is coding theory?



Shannon's communication diagram

What is coding theory?



Shannon's communication diagram

Formal definitions of codes and weights

Linear $[n, k]$ code	Linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.

Formal definitions of codes and weights

Linear $[n, k]$ code	Linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.
Generator matrix	The rows of this $k \times n$ matrix form a basis for C.

Formal definitions of codes and weights

Linear $[n, k]$ code Linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.

Generator matrix The rows of this $k \times n$ matrix form a basis for C.

Support The coordinates of a word which are nonzero.

Formal definitions of codes and weights

Linear $[n, k]$ code Linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.

Generator matrix The rows of this $k \times n$ matrix form a basis for C.

Support The coordinates of a word which are nonzero.

Weight The number of nonzero coordinates of a word, i.e. the size of the support.

Formal definitions of codes and weights

Linear $[n, k]$ code	Linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.
Generator matrix	The rows of this $k \times n$ matrix form a basis for C.
Support	The coordinates of a word which are nonzero.
Weight	The number of nonzero coordinates of a word, i.e. the size of the support.

Weight enumerator

The homogeneous polynomial counting the number of words of a given weight, notation:

$$W_C(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w.$$

Example

The $[7,4]$ Hamming code over $\mathbb{F}_2$ has generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The weight enumerator is equal to

$$W_C(X,Y) = X^7 + 7X^4Y^3 + 7X^3Y^4 + Y^7.$$

Generalized weight enumerator

For a subcode $D \subseteq C$ we define

Support Union of the support of all words in D, i.e. all coordinates which are not always zero.

Weight Size of the support.

Generalized weight enumerator

For a subcode $D \subseteq C$ we define

Support Union of the support of all words in D, i.e. all coordinates which are not always zero.

Weight Size of the support.

Generalized weight enumerators

The homogeneous polynomials counting for each dimension $r = 0, \ldots, k$ the number of subcodes of a given weight, notation:

$$W_C^r(X, Y) = \sum_{w=0}^{n} A_w^r X^{n-w} Y^w$$

Example

The $[7,4]$ Hamming code has generalized weight enumerators

$$\begin{aligned}
W_C^0(X,Y) &= X^7 \\
W_C^1(X,Y) &= 7X^4Y^3 + 7X^3Y^4 + Y^7 \\
W_C^2(X,Y) &= 21X^2Y^5 + 7XY^6 + 7Y^7 \\
W_C^3(X,Y) &= 7XY^6 + 8Y^7 \\
W_C^4(X,Y) &= Y^7
\end{aligned}$$

Extended weight enumerator

Extension code Code over some extensionfield $\mathbb{F}_{q^m}$ having the same generator matrix as C, notation: $C \otimes \mathbb{F}_{q^m}$.

Extension code Code over some extensionfield $\mathbb{F}_{q^m}$ having the same generator matrix as C, notation: $C \otimes \mathbb{F}_{q^m}$.

Extended weight enumerator

The polynomial "counting the number of words in an extension code", notation:

$$W_C(X,Y,T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.$$

Note that with $T = q^m$ we have $W_C(X,Y,q^m) = W_{C\otimes\mathbb{F}_{q^m}}(X,Y)$.

Example

The $[7,4]$ Hamming code has extended weight enumerator

$$\begin{aligned} W_C(X,Y,T) &= X^7 + \\ &\quad 7(T-1)X^4Y^3 + \\ &\quad 7(T-1)X^3Y^4 + \\ &\quad 21(T-1)(T-2)X^2Y^5 + \\ &\quad 7(T-1)(T-2)(T-3)XY^6 + \\ &\quad (T-1)(T^3-6T^2+15T-13)Y^7 \end{aligned}$$

Matroids

Matroid theory generalizes the notion of "linear independence".

- Vector space: linear independent vectors, basis
- Graph: tree, minimal spanning tree
- Matroid: independent set, basis

A matroid consist of a finite set E and a set of independent sets from 2^E having some defining properties.

Matroids

Matroid theory generalizes the notion of "linear independence".

- Vector space: linear independent vectors, basis
- Graph: tree, minimal spanning tree
- Matroid: independent set, basis

A matroid consist of a finite set E and a set of independent sets from 2^E having some defining properties.

Example

A code can be viewed as a matroid by considering the columns of a generator matrix and their dependance in $\mathbb{F}_q^k$.

A matroid has a *rank function*, notation $r(A)$, associating a non-negative integer to every subset A of E.

Example

For matroid from a generator matrix G of a code, $r(A)$ is the rank of the submatrix formed by the columns of G indexed by A. Furthermore, $r(E) = k$.

Tutte polynomial

A matroid has a *rank function*, notation $r(A)$, associating a non-negative integer to every subset A of E.

Example

For matroid from a generator matrix G of a code, $r(A)$ is the rank of the submatrix formed by the columns of G indexed by A. Furthermore, $r(E) = k$.

Tutte polynomial

The Tutte polynomial is defined by

$$t_G(X, Y) = \sum_{A \subseteq E} (X-1)^{r(E)-r(A)} (Y-1)^{|A|-r(A)}.$$

Connections – some formulas

We can write the extended weight enumerator in terms of the generalized weight enumerator:

$$W_C(X,Y,T) = \sum_{r=0}^{k} \left(\prod_{j=0}^{r-1} (T - q^j) \right) W_C^r(X,Y);$$

and in terms of the Tutte polynomial:

$$W_C(X,Y,T) = (X-Y)^k Y^{n-k} \; t_G \left(\frac{X + (T-1)Y}{X-Y}, \frac{X}{Y} \right).$$

Connections – overview

$$
\begin{tikzcd}
W_C(X,Y) & & & \\
& & W_C(X,Y,T) \arrow[llu] \arrow[ld, shift left] \arrow[d, shift left] \arrow[rdd, leftrightarrow] & \\
& \{W_C^r(X,Y)\}_{r=0}^k \arrow[luu] \arrow[ru, shift left] \arrow[r, shift left] \arrow[rrd, leftrightarrow] & t_G(X,Y) \arrow[l, shift left] \arrow[u, shift left] & \\
& & & \{W_C^r(X,Y,T)\}_{r=0}^k
\end{tikzcd}
$$

Further work

- Connections with other classifying polynomials:
  - Codes and zeta-function
  - Arrangement of hyperplanes and Poincaré polynomial
  - Arrangement of hyperplanes and zeta-function
  - Lattices and theta-function
- Concrete computations for special classes of codes
  - Cyclic codes
  - Algebraic geometry codes
- Characterization of extended weight enumerator

Further work

- Extend known theory to extended weight enumerator:
  - Gleason's theory for self-dual codes
  - Codes over rings
- Probability of correct decoding
- Complexity issues