Classifying polynomials of linear codes and hyperplane arrangements

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Outline

Extended weight enumerator

Extension code $[n, k]$ code over some extension field $\mathbb{F}_{q^m}$ generated by the words of C, notation: $C \otimes \mathbb{F}_{q^m}$.

Generator matrix All the extension codes of C have the same generator matrix G.

Extended weight enumerator

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Extended weight enumerator

The homogeneous polynomial counting the number of words of a given weight "for all extension codes", notation:

$$W_C(X,Y,T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.$$

Note that with $T = q^m$ we have $W_C(X,Y,q^m) = W_{C\otimes\mathbb{F}_{q^m}}(X,Y)$.

Hyperplane arrangements

Arrangement of hyperplanes	n-tuple of hyperplanes in $\mathbb{F}_q^k$.
Essential arrangement	Intersection of all hyperplanes is $\{\mathbf{0}\}$, hyperplanes are in $\mathbb{P}^{k-1}(\mathbb{F}_q)$.

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Columns of a generator matrix G of a linear $[n, k]$ code form a hyperplane arrangement. Notation: $(H_1, \ldots, H_n)$.

- One-to-one correspondence between equivalence classes.
- Independent of choice of G, so notation: $\mathcal{A}_C$.
- Also valid over an extension field $\mathbb{F}_{q^m}$.

Hyperplane arrangements

Theorem

Let C be a linear $[n, k]$ code with generator matrix G and $\mathbf{x} \in \mathbb{F}_q^k$. Then for every word $\mathbf{c} = \mathbf{x}G$ we have that $n - \mathrm{wt}(\mathbf{c})$ is equal to the number of hyperpanes in $\mathcal{A}_C$ through $\mathbf{x}$.

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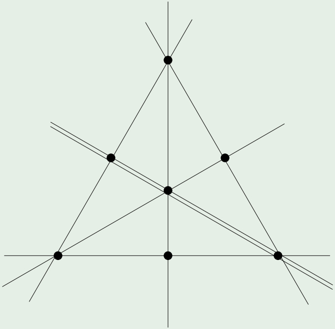
To find the extended weight enumerator of C, we have to look at the intersections of the hyperplanes in the arrangement $\mathcal{A}_C$.

Determination of extended weight enumerator

Example



Let $q > 2$ and C generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Use that $A_w(T)$ is the number of points on $n - w$ hyperplanes.

The extended weights are given by

$$A_0(T) = 1$$

The zero word is on all hyperplanes.

Determination of extended weight enumerator

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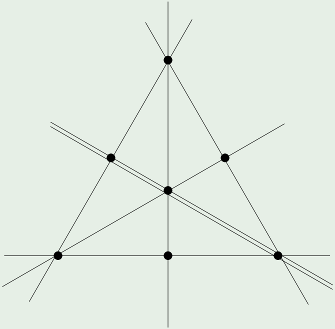

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The extended weights are given by

$$A_1(T) = A_2(T) = 0$$

No points are on 6 or 5 hyperplanes.

Determination of extended weight enumerator

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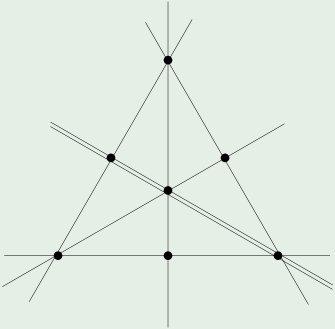

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Use that $A_w(T)$ is the number of points on $n - w$ hyperplanes.

The extended weights are given by

$$A_3(T) = 2(T - 1)$$

Two projective points are on 4 hyperplanes.

Determination of extended weight enumerator

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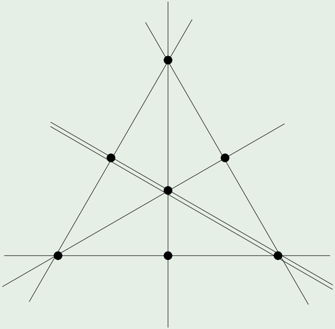

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Use that $A_w(T)$ is the number of points on $n - w$ hyperplanes.

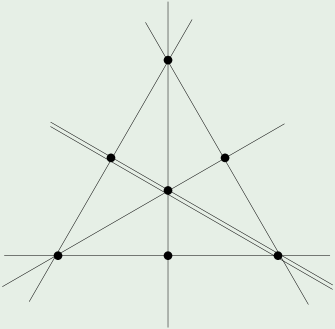
The extended weights are given by

$$A_4(T) = 3(T - 1)$$

Three projective points are on 3 hyperplanes.

Determination of extended weight enumerator

Example



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Use that $A_w(T)$ is the number of points on $n-w$ hyperplanes.

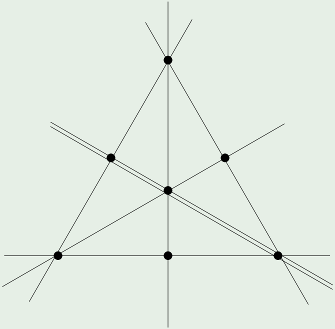
The extended weights are given by

$$A_5(T) = T(T-1)$$

$(T+1)-3$ points on double line; two points on 2 hyperplanes.

Determination of extended weight enumerator

Example



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Use that $A_w(T)$ is the number of points on $n - w$ hyperplanes.

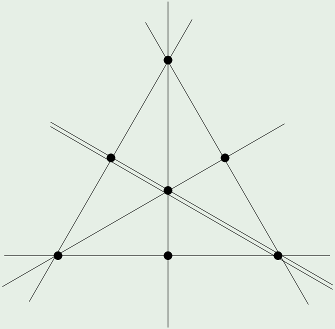
The extended weights are given by

$$A_6(T) = 5(T-1)(T-2)$$

$(T+1) - 3$ extra points on 5 projective lines.

Determination of extended weight enumerator

Example



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Use that $A_w(T)$ is the number of points on $n - w$ hyperplanes.

The extended weights are given by

$$A_7(T) = (T-1)(T^2 - 5T + 6)$$

The total number of projective points is $T^2 + T + 1$.

Determination of extended weight enumerator

For all subsets $J \subseteq [n]$ define

$$\begin{array}{rcl} C(J) & = & \{\mathbf{c} \in C : c_j = 0 \text{ for all } j \in J\} \\ l(J) & = & \dim C(J) \\ B_J(T) & = & T^{l(J)} - 1 \\ B_t(T) & = & \sum\limits_{|J|=t} B_J(T) \end{array}$$

Note:

- $\dim C(J) = \dim (C \otimes \mathbb{F}_{q^m})(J)$.
- $B_J(q^m)$ is the number of nonzero words in $(C \otimes \mathbb{F}_{q^m})(J)$.
- $C(J) \cong \cup_{j \in J} H_j$, where H_j are in $\mathcal{A}_C$.

Determination of extended weight enumerator

To determine the number of points on $n - w$ hyperplanes, we use an inclusion-exclusion argument.

Proposition

$$A_w(T) = \sum_{t=n-w}^{n} (-1)^{n+w+t} \binom{t}{n-w} B_t(T)$$

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Theorem

The extended weight enumerator can be written as

$$W_C(X, Y, T) = X^n + \sum_{t=0}^{n} B_t(T)(X - Y)^t Y^{n-t}.$$

Geometric lattices

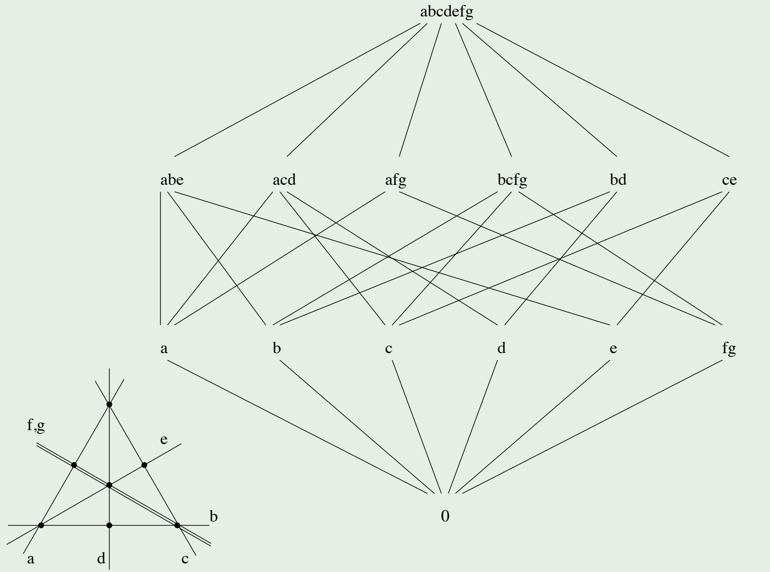
A *geometric lattice* L is a set with partial ordering $\leq$ and some additional specifying properties.

An arrangement $\mathcal{A}_C$ gives rise to a geometric lattice $L(C)$:

Elements	All intersections of hyperplanes
Ordering	$x \leq y$ if $y \subseteq x$
Minimum	Whole space $\mathbb{F}_q^k$
Maximum	Zero vector $\mathbf{0} \in \mathbb{F}_q^k$
Rank	Corank of x in $\mathbb{F}_q^k$
Atoms	The hyperplanes of the arrangements, without multiplicity

Geometric lattices

Example



Coboundary polynomial

The *Möbius function* of a geometric lattice is defined for all $x \leq y$ by $\mu_L(x, x) = 0$ and

$$\sum_{x \leq z \leq y} \mu_L(x, z) = \sum_{x \leq z \leq y} \mu_L(z, y) = 0.$$

Note the function is alternating in the rank of the geometric lattice.

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Coboundary polynomial

The coboundary of a geometric lattice is defined by

$$\chi_L(S, T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu_L(x, y) S^{|x|} T^{r(L) - r(y)}$$

where $|x|$ is the number of atoms smaller then x.

Coboundary polynomial

If all the hyperplanes in $\mathcal{A}_C$ are distinct, we say the code is *projective*. (Equivalent: $d(C^\perp) \geq 3$.)

Theorem

The extended weight enumerator of a projective code is determined the coboundary polynomial of the associated geometric lattice, and vise versa, via

$$\begin{aligned} \chi_{L(C)}(S,T) &= W_C(S,1,T) \\ W_C(X,Y,T) &= Y^n \cdot \chi_{L(C)}(XY^{-1},T) \end{aligned}$$

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- Extending the underlying field gives extension codes $C \otimes \mathbb{F}_{q^m}$, and we define the extended weight eumerator $W_C(X, Y, T)$.

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- By viewing the colums of G as hyperplanes, we associate an arrangement $\mathcal{A}_C$ to a code.
- Finding the extended weight enumerator means counting points in intersections of hyperplanes.
- This counting can also be done using the geometric lattice associated with the arrangement.
- For projective codes, the coboundary polynomial of $L(C)$ is equivalent to the extended weight enumerator.