Truncation formulas for invariant polynomials of matroids and geometric lattices

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Outline

Geometric lattices

A *geometric lattice* L is a set with partial ordering $\leq$ and some additional specifying properties.

A matroid M with ground set E gives rise to a geometric lattice $L(M)$, called the *lattice of flats*:

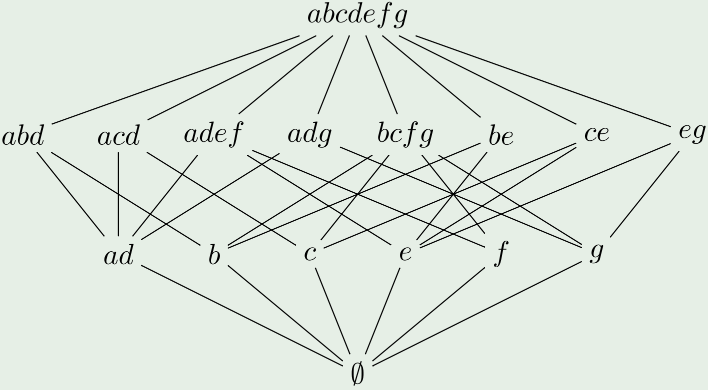
elements	all flats of M
ordering	$x \leq y$ if $x \subseteq y$
minimum	empty set $\emptyset$
maximum	whole ground set E
rank	rank of the flat x in M
atoms	all flats of rank 1

If the matroid is *simple*, $L(M)$ is equivalent to M.

Geometric lattices

Example

$$\begin{pmatrix} a & b & c & d & e & f & g \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & \alpha \end{pmatrix}$$



Möbius function

The *Möbius function* of a geometric lattice is defined for all $x \leq y$ by $\mu_L(x,x) = 1$ and

$$\sum_{x \leq z \leq y} \mu_L(x,z) = \sum_{x \leq z \leq y} \mu_L(z,y) = 0.$$

If x and y are not comparable, then $\mu_L(x,y) = 0$.

Note the function is alternating in the rank of the geometric lattice.

Truncation

Idea: "cutting off elements of highest rank"

Truncated matroid $T(M)$

Several equivalent descriptions:

independent sets	all the independent sets of M of rank $< r$
bases	independent sets of rank $r-1$ in M
rank function	$r_{T(M)}(A) = \min\{r_M(A), r-1\}$

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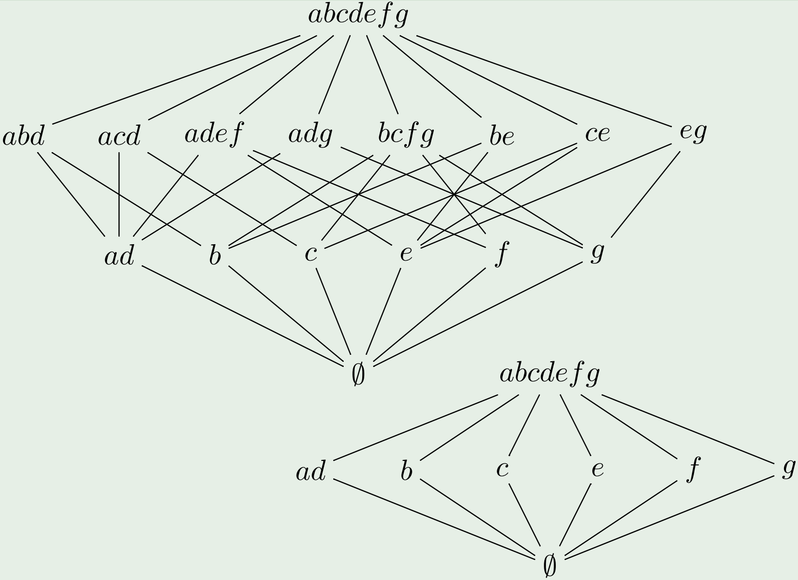
Truncated lattice $T(L)$

Remove all elements of rank $r-1$ and preserve partial ordering.

Truncation

Example



Truncation formulas

Various invariant polynomials are associated with matroids and geometric lattices:

- rank generating polynomial
- coboundary polynomal
- Möbius polynomial
- (spectrum polynomial)

Question: do these polynomials determine the polynomials associated with $T(M)$?

Answer: yes!

Truncation formulas

Rank generating function

The rank generating function of a matroid is defined by

$$R_M(X,Y) = \sum_{A \subseteq E} X^{r(E)-r(A)} Y^{|A|-r(A)}.$$

Truncation formulas

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Theorem (Britz, 2007)

Let M be a matroid. Then

$$X \cdot R_{T(M)}(X,Y) = R_M(X,Y) + (XY-1) \cdot R_M(0,Y).$$

Truncation formulas

Coboundary polynomial

The coboundary polynomial of a geometric lattice is defined by

$$\chi_L(S,T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu_L(x,y)\, S^{a_L(x)} T^{r(L)-r(y)}$$

where $a_L(x)$ is the number of atoms a in L such that $a \leq x$.

Truncation formulas

Coboundary polynomial

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Theorem (Crapo, 1968)

The coboundary polynomial of a lattice of flats is determined by the rank generating function via

$$\chi_{L(M)}(S,T) = (S-1)^{r(M)} \cdot R_M\left(\frac{T}{S-1}, S-1\right).$$

Truncation formulas

Theorem

Let L be a geometric lattice of rank $r \geq 3$. Then

$$T \cdot \chi_{T(L)}(S,T) = \chi_L(S,T) + (T-1) \cdot \chi_L(S,0).$$

Proof:

- use relation with rank generating function; or:
- use induction formula for Möbius function to write both sides in terms of elements with low rank.

Truncation formulas

Möbius polynomial

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Truncation formulas

Möbius polynomial

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$$\mu_L(S,T) = \sum_{x \in L} \sum_{x \le y \in L} \mu_L(x,y)\, S^{r_L(x)} T^{r(L) - r_L(y)}.$$

Theorem

Let L be a geometric lattice of rank $r > 0$. Then

$$T \cdot \mu_{T(L)}(S,T) = \mu_L(S,T) + (T-1) \cdot \mu_L(S,0) + S^{r-1}T - S^r T.$$

Representation of truncated matroid

Theorem (Brylawski, 1986)

Let M be a representable matroid. Then $T(M)$ is representable over a transcendental extension field.

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Theorem

- *If M is representable over an infinite field, then $T(M)$ is representable over the same field.*
- *If M is representable over a finite field $\mathbb{F}_q$, then $T(M)$ is representable over $\mathbb{F}_{q^m}$ with $m \geq \lceil \log_q \binom{n}{r(M)-1} \rceil + 1$.*

So there are linear codes associated to truncated matroids.

Extended weight enumerator

Extension code $[n, k]$ code over some extension field $\mathbb{F}_{q^m}$ generated by the words of C, notation: $C \otimes \mathbb{F}_{q^m}$.

Generator matrix All the extension codes of C have the same generator matrix G.

Extended weight enumerator

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Extended weight enumerator

The homogeneous polynomial counting the number of words of a given weight "for all extension codes", notation:

$$W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.$$

Note that with $T = q^m$ we have $W_C(X, Y, q^m) = W_{C \otimes \mathbb{F}_{q^m}}(X, Y)$.

Extended weight enumerator

- The rank generating function completely determines the extended weight enumerator, and vice versa.
- If M is representable over multiple fields, all corresponding codes have same extended weight enumerator $W_M(X, Y, T)$.

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- If M is representable over multiple fields, all corresponding codes have same extended weight enumerator $W_M(X, Y, T)$.

Theorem

Let M be a matroid. Then for all codes determined by M we have

$$T \cdot W_{T(M)}(X, Y, T) = W_M(X, Y, T) + (T - 1) \cdot W_M(X, Y, 0).$$

Extended weight enumerator

Example

M represented by $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & \alpha \end{pmatrix}$ over $\mathbb{F}_q, \ q > 2$

$$\begin{aligned} W_M(X,Y,T) \ = \ & X^7 + \\ & 2(T-1)X^4Y^3 + \\ & 3(T-1)X^3Y^4 + \\ & T(T-1)X^2Y^5 + \\ & (T-1)(T-2)(T-3)Y^7 \end{aligned}$$

Extended weight enumerator

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$$\begin{aligned} W_{T(M)}(X,Y,T) \quad = \quad & X^7 + \\ & (T-1)X^2Y^5 + \\ & 5(T-1)XY^6 + \\ & (T-1)(T-5)Y^7 \end{aligned}$$

Extended weight enumerator

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M represented by $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & \alpha \end{pmatrix}$ over $\mathbb{F}_q,\ q > 2$

$$\begin{aligned} W_{T(M)}(X,Y,T) &= X^7 + \\ &\quad (T-1)X^2Y^5 + \\ &\quad 5(T-1)XY^6 + \\ &\quad (T-1)(T-5)Y^7 \end{aligned}$$

$T(M)$ represented by $\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 3 & 4 \end{pmatrix}$ over $\mathbb{F}_5$

Further questions

- Better bounds for representation of $T(M)$ via codes?
- Formulas for principal truncation, Dilworth truncation
- Connections to Duursma zeta functions
- Möbius polynomial: unimodal conjecture on the Whitney numbers
- Spectrum polynomial: does it determine rank generating function?

Thank you for your attention.