# Weight enumeration of codes from finite spaces

Relinde Jurrius

Eindhoven University of Technology, The Netherlands

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# Outline

Projective systems and weight enumeration

Generalized weight enumerator

Codes form finite spaces

- Finite projective space: simplex code
- Finite affine space: 1-st order Reed-Muller code

Extended weight enumerator

Further questions and applications

# Projective systems

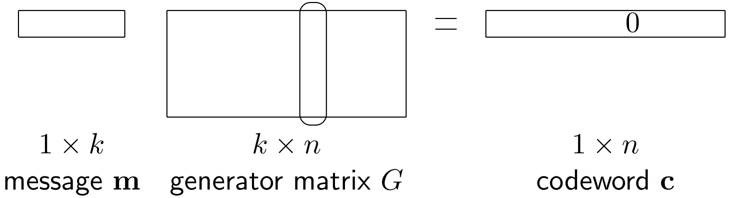
projective system $n$-tuple $\mathcal{P} = (P_1, \ldots, P_n)$

points $P_j \in \mathrm{PG}(r, q)$ in general position

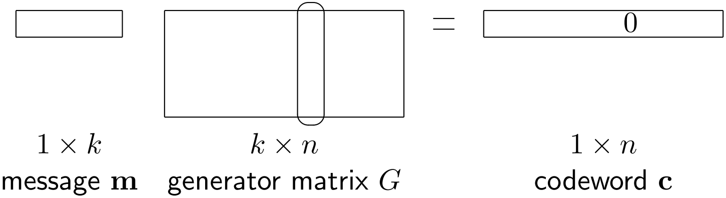
Matrix $G_{\mathcal{P}}$ with coordinates of points of $\mathcal{P}$ as columns generates linear $[n, r+1]$ code.

$$\left\{ \begin{array}{c} \text{equiv. classes of} \\ \text{linear } [n,k] \text{ codes} \\ \text{over } \mathrm{GF}(q) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{equiv. classes of} \\ \text{proj. systems of length } n \\ \text{over } \mathrm{PG}(k-1,q) \end{array} \right\}$$

# Weights in linear codes



# Weights in linear codes




**Theorem**

$$c_j = 0 \iff P_j \textit{ is in nullspace of } \mathbf{m}$$

We can determine weights by counting points $P_j$ on hyperplanes.

# Generalized weight enumerator

For a subcode $D \subseteq C$ we define

support $\mathrm{supp}(D)$ union of the support of all words in $D$

zero set $\mathrm{zero}(D)$ complement of support, i.e., all coordinates that are always zero

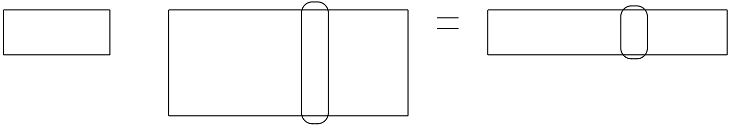
weight $\mathrm{wt}(D)$ size of the support

## Generalized weight enumerators

Polynomials counting for every dimension the number of subcodes of a given weight:

$$W_C^{(r)}(X,Y) = \sum_{\substack{D \in C \\ \dim(D)=r}} X^{|\mathrm{zero}(D)|} Y^{|\mathrm{supp}(D)|}$$

# Weight enumeration of subcodes



$r \times k$ nullspace $= \Pi$ — $k \times n$ generator matrix $G$ — $r \times n$ generates $D$

$$\begin{array}{c} \Pi \subseteq \mathrm{PG}(k-1,q) \\ \mathrm{codim}(\Pi) = r \end{array} \quad \longleftrightarrow \quad \begin{array}{c} D \subseteq C \\ \dim(D) = r \end{array}$$

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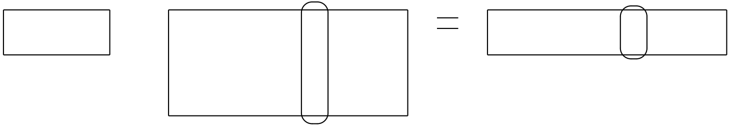

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**Theorem**

$$j \in \operatorname{zero}(D) \iff P_j \in \Pi$$

# Codes from finite spaces

Let $\mathcal{P}$ contain all points in $\mathrm{PG}(s-1,q)$. The corresponding code is the simplex code $\mathcal{S}_q(s)$. It has length $\dfrac{q^s-1}{q-1}$ and dimension $s$.

**Example**

$\mathcal{S}_2(3)$ has generator matrix

$$G_{\mathcal{P}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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- $j \in \mathrm{zero}(D) \iff P_j \in \Pi$
- $\mathcal{P}$ contains all points in $\mathrm{PG}(s-1,q)$
- $|\mathrm{zero}(D)| = |\Pi|$ for all $D$

**Theorem**

$$W^{(r)}_{\mathcal{S}_q(s)}(X,Y) = \begin{bmatrix} s \\ r \end{bmatrix}_q X^{(q^{s-r}-1)/(q-1)} Y^{(q^s-q^{s-r})/(q-1)}$$

# Codes from finite spaces

Let $\mathcal{P}$ contain all points in $\mathrm{AG}(s-1,q)$, viewed as all points in $\mathrm{PG}(s-1,q)$ not on a hyperplane $H$. The corresponding code is the 1-st order Reed Muller code $\mathcal{RM}_q(1,s-1)$. It has length $q^{s-1}$ and dimension $s$.

## Example

$\mathcal{RM}_2(1,3)$ has generator matrix

$$G_{\mathcal{P}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

# Codes from finite spaces

$$\begin{matrix} \Pi \subseteq \mathrm{PG}(k-1,q) \\ \mathrm{codim}(\Pi) = r \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} D \subseteq C \\ \dim(D) = r \end{matrix}$$

- $j \in \mathrm{zero}(D) \iff P_j \in \Pi$
- if $\Pi \subseteq H$, no $P_j$ is in $\Pi$, so $\mathrm{wt}(D) = n$
- if $\Pi \not\subseteq H$, all of the $P_j$ in $\Pi$ form a subspace of $\mathrm{AG}(s-1,q)$ of codimension $r$

**Theorem**

$$W^{(r)}_{\mathcal{RM}_q(1,s-1)}(X,Y) = \begin{bmatrix} s-1 \\ r-1 \end{bmatrix}_q Y^n + q^r \begin{bmatrix} s-1 \\ r \end{bmatrix}_q X^{q^{s-1-r}} Y^{q^{s-1}-q^{s-1-r}}$$

# Extended weight enumerator

For every linear $[n, k]$ code $C$ with generator matrix $G$ we have:

| | |
|---|---|
| Extension code | $[n, k]$ code $C \otimes \mathrm{GF}(q^m)$ over some extension field $\mathrm{GF}(q^m)$ generated by the words of $C$. |
| Generator matrix | All the extension codes of $C$ have the same generator matrix $G$. |

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**Generator matrix** All the extension codes of $C$ have the same generator matrix $G$.

**Extended weight enumerator**

Polynomial counting "for all extension codes" the number of words of a given weight:

$$W_C(X,Y,T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.$$

So for $T = q^m$ we have $W_C(X,Y,q^m) = W_{C\otimes \mathrm{GF}(q^m)}(X,Y)$.

# Extended weight enumerator

The etended weight enumerator is completely determined by the set of generalized weight enumerators (and vice versa):

**Theorem**

$$W_C(X,Y,T) = \sum_{r=0}^{k} \left( \prod_{j=0}^{r-1} (T - q^j) \right) W_C^{(r)}(X,Y).$$

Moreover, their sets of supports are the same.

# Supports

**Theorem (Simplex code)**

*Let* $\mathbf{c} \in \mathcal{S}_q(s) \otimes \mathrm{GF}(q^m)$ *with* $\mathrm{wt}(\mathbf{c}) = (q^s - q^{s-r})/(q-1)$, $r < m$. *Then the points in* $\mathcal{P}$ *indexed by* $\mathrm{zero}(\mathbf{c})$ *are all the points in a subspace of* $\mathrm{PG}(s-1, q)$ *of codimension* $r$.

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**Theorem (1-st order Reed-Muller code)**

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So: the sets $\mathrm{zero}(\mathbf{c})$ for all codewords contains the incidence design of a finite geometry.

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- Link with perfect matroid designs and associated polynomials
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- Dimension of a design (generalization of $p$-rank)

Thank you for your attention.