Deletion and contraction in q-matroids

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q-Analogues

Finite set $\longrightarrow$ finite vectorspace over $\mathbb{F}_q$

Example

$$\binom{n}{k} = \text{number of sets of size } k \text{ contained in set of size } n$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \text{number of } k\text{-dim subspaces of } n\text{-dim vectorspace over } \mathbb{F}_q$$

$$= \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}$$

q-Analogues

Example

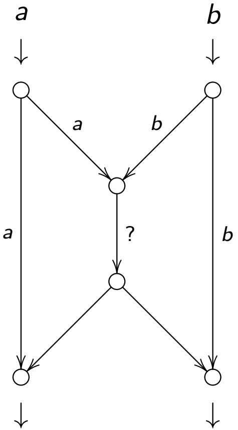
t-(v, k, λ) design: pair $(X, \mathcal{B})$ with

- X set with v elements (points)
- $\mathcal{B}$ family of subsets of X of size k (blocks)
- Every t-tuple of points is contained in exactly λ blocks

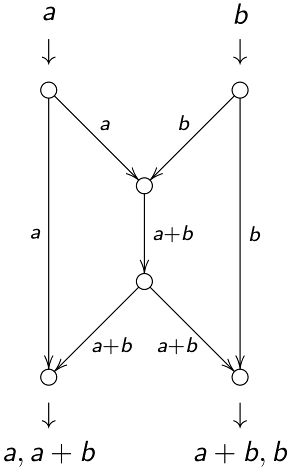
t-$(v, k, \lambda; q)$ q-design: pair $(X, \mathcal{B})$ with

- X v-dim vectorspace over $\mathbb{F}_q$
- $\mathcal{B}$ family of k-dim subspaces of X (blocks)
- Every t-dim subspace is contained in exactly λ blocks

Motivation: network coding



Motivation: network coding



Idea: send (rows of) matrices instead of vectors

Better idea: send (bases of) subspaces instead of matrices

Motivation: network coding

Codewords are vectors:

'Ordinary' error-correcting codes

Codewords are matrices:

Rank metric codes

q-analogue of 'ordinary' codes

Codewords are subspaces:

Subspace codes

Constant dimension, constant weight: q-design

q-Analogues

finite set	finite space $\mathbb{F}_q^n$
element	1-dim subspace
size	dimension
n	$\frac{q^n-1}{q-1}$
intersection	intersection
union	sum
complement	quotient space

From q-analogue to 'normal': let $q \to 1$.

Matroids and q-matroids

Matroid: a pair $(E, \mathcal{I})$ with

- E finite set;
- $\mathcal{I} \subseteq 2^E$ family of subsets of E, the *independent sets*, with:
  - (I1) $\emptyset \in \mathcal{I}$
  - (I2) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
  - (I3) If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

Examples:

- Set of vectors; independence $=$ linear independence
- Set of edges of a graph; independence $=$ cycle free

Matroids and q-matroids

q-Matroid: a pair $(E, \mathcal{I})$ with

- E finite space;
- $\mathcal{I} \subseteq 2^E$ family of subspaces of E, the *independent spaces*, with:
  (I1) $\mathbf{0} \in \mathcal{I}$
  (I2) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
  (I3) If $A, B \in \mathcal{I}$ and $\dim A > \dim B$ then there is a 1-dim subspace $a \subseteq A$, $a \not\subseteq B$ such that $B + a \in \mathcal{I}$.

Why study q-matroids?

Matroids generalize:

- codes
- graphs
- some designs

q-Matroids generalize:

- rank metric codes
- q-graphs ?
- q-designs ?

Deletion / contraction

$M = (E, \mathcal{I})$ matroid, e element of E

Deletion $M - e$

Ground set: $E - \{e\}$

Independent sets: members of $\mathcal{I}$ that do not contain e

$= \{I \in \mathcal{I} : e \notin I\}$

Contraction M/e

Ground set: $E - \{e\}$

Independent sets: members of $\mathcal{I}$ containing e, with e removed

$= \{I - \{e\} : I \in \mathcal{I}, e \in I\}$

Deletion and contraction

$M = (E, \mathcal{I})$ q-matroid, e 1-dim subspace of E

Deletion $M - e$

Ground space: E/e with ‘projection’ $\pi : E \to E/e$

Independent spaces: members of $\mathcal{I}$ that do not contain e, mapped to E/e

$= \{\pi(I) : I \in \mathcal{I}, e \not\subseteq I\}$

Contraction M/e

Ground space: E/e with ‘projection’ $\pi : E \to E/e$

Independent spaces: members of $\mathcal{I}$ containing e, mapped to E/e

$= \{\pi(I) : I \in \mathcal{I}, e \subseteq I\}$

Deletion and contraction

e element of finite set E

$\{\text{subsets not containing } e\} \cup \{\text{subsets containing } e\} = 2^E$

$|\mathcal{I}(M - e)| + |\mathcal{I}(M/e)| = |\mathcal{I}(M)|$

e 1-dimensional subspace of $E = \mathbb{F}_q^n$

If $A, B \in \mathcal{I}$ do not contain e, it is possible that $\pi(A) = \pi(B)$.

How often does that happen?

Deletion and contraction

Total number of k-dim subspaces in E: $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Number of k-dim subspaces containing e: $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$

Number of k-dim subspaces not containing e:
$\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$

$\Longrightarrow$ k-dim space in E/e has q^k pre-images in E not containing e

$$\sum_{k=0}^{n-1} q^k |\{I : I \in \mathcal{I}(M-e), \dim I = k\}| + |\mathcal{I}(M/e)| = |\mathcal{I}(M)|$$

Overview

- q-Analogues attract attention nowadays because of network coding.
- We should study q-matroids for the same reasons we study matroids: they generalize several discrete structures.
- Deletion and contraction for q-matroids is defined.
- Counting on deletion and contraction in q-matroids gives extra factors q^k. This needs better notation.
- To do: deletion and contraction for the q-rank generating function and q-Tutte polynomial.
- To do: link with puncturing/shortening in rank metric codes, derived/residual q-design.

Thank you for your attention.