A combinatorial view on derived codes

Relinde Jurrius
(joint work with Philippe Cara)

Vrije Universiteit Brussel, Belgium
$\rightarrow$ University of Neuchâtel, Switzerland

Finite Geometries
September 16, 2014

Codes and lattices

Linear code	k-dim subspace of $\mathrm{GF}(q)^n$
Arrangement of hyperplanes	n-tuple of hyperplanes in $\mathrm{GF}(q)^k$
Projective system	n-tuple of points in $\mathrm{PG}(k-1, q)$.

- One-to-one correspondence between equivalence classes.
- Independent of choice of generator matrix, so notation: $\mathcal{A}_C$ or $\mathcal{P}_C$.

Codes and lattices

Lattice: poset with *join* (smallest upper boud), $x \vee y$
and *meet* (greatest lower bound) $x \wedge y$

Geometric lattice:

- atomic: every element is join of rank-1 elements (*atoms*)
- semimodular: rank function with $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$
- no infinite chains

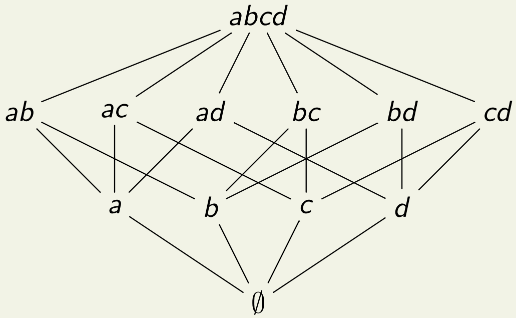
Codes and lattices

Projective system: atoms = points, elements = spans

Hyperplane arrangement: atoms = hyperplanes, elements = intersections

Example

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$



Derived code

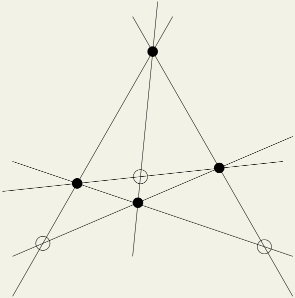
- Start with $[n, k]$ code.
- Consider the projective system $\mathcal{P}_C$.
- Look at all hyperplanes spanned by $k - 1$ points of $\mathcal{P}_C$.
  (Ignore $k - 1$ points that span spaces of lower dimension.)
- Remove (multiple) copies of hyperplanes.
- These hyperplanes form an arrangement $\mathcal{A}$.
- The *derived code* $D(C)$ is the code such that $\mathcal{A} = \mathcal{A}_{D(C)}$.

Derived code

Example

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

C $\qquad$ $D(C)$



Motivation

The derived code was introduced in the study of the *coset leader* and *list weight enumerator*.

The coset leader weight enumerator is interesting because:

- Determines the probability of correct decoding in coset leader decoding.
- Determines the average of changed symbols in *steganography* (information hiding).

The list weight enumerator is interesting because:

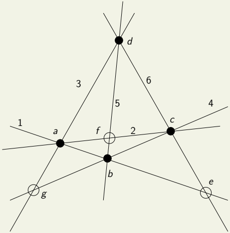
- Determines the size of lists in list decoding.
- Determines the probability of correct decoding in list decoding.

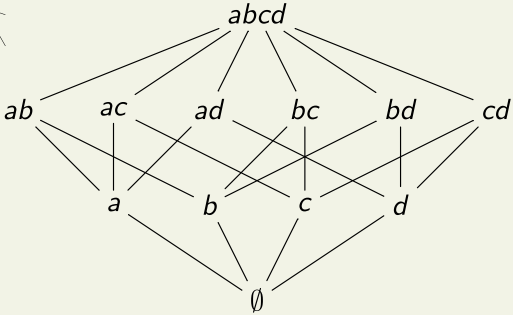
Derived code

From a lattice point of view, this is how we make a derived code:

- Start with $[n, k]$ code.
- Consider the geometric lattice $\mathcal{L}$ of $\mathcal{P}_C$.
- Turn it upside down: $\mathcal{L}^{op}$.
- Add extra elements above atom level such that
  - $\mathcal{L}^{op} \hookrightarrow \mathcal{L}(D(C))$ in the "right" way;
  - $\mathcal{L}(D(C))$ is a geometric lattice.

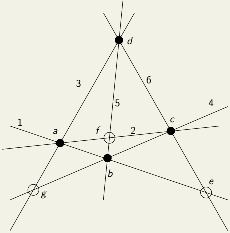
Example

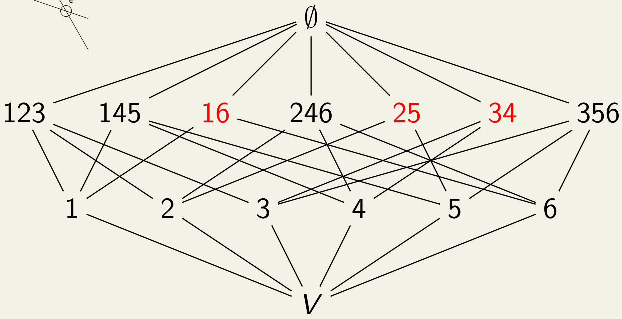




Derived code

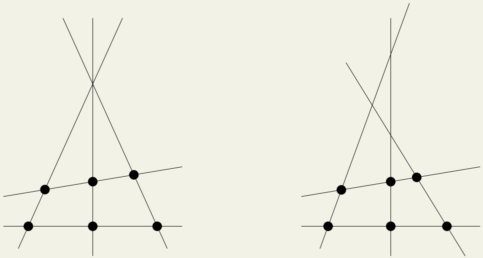
Example





Derived code

Example



Codes with equal geometric lattices may have different derived codes!

Results

We answered two open questions:

- When is $C \cong D(C)$?
- Can we define a *derived lattice*, by taking the derived arrangement "as general as possible"?

When is $C \cong D(C)$?

We need: $\mathcal{L}(C) = \mathcal{L}(C)^{op} = \mathcal{L}(D(C))$ and $\mathcal{L}^{op}$ is a geometric lattice.

Theorem

The following are equivalent:

- $C \cong D(C)$
- $r(x \vee y) + r(x \wedge y) = r(x) + r(y)$
- $\mathcal{P}_C$ *contains all points of some* $\mathrm{PG}(k-1, q)$
- C *is the q-ary simplex code*

Derived lattice

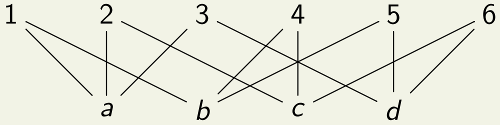
How to create the *derived lattice* $D(\mathcal{L})$:

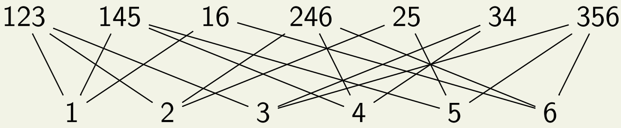
- Start with a geometric lattice $\mathcal{L}$.
- The atoms of $D(\mathcal{L})$ are the co-atoms of $\mathcal{L}$.
- For all subsets I of atoms:
  - If $r(\mathcal{L}) - r(\bigwedge I) \leq |I|$ in $\mathcal{L}$, then $\bigvee I = (\bigwedge I)^{op}$ in $D(\mathcal{L})$.
  - If $r(\mathcal{L}) - r(\bigwedge I) > |I|$ in $\mathcal{L}$, then $\bigvee I$ is a new element in $D(\mathcal{L})$ with $r^*(\bigvee I) = |I|$.
- Partial ordering: $\bigvee I \leq \bigvee J$ iff $I \subseteq J$.

Rank function: $r^*(\bigvee I) = \min\{r(\mathcal{L}) - r(\bigwedge I), |I|\}$

Derived lattice

Example





$I = \{1, 2\}$
$3 - r(a) = 3 - 1 = 2$
$2 \leq 2$
$\rightarrow$ no new element

$I = \{1, 6\}$
$3 - r(0) = 3 - 0 = 3$
$3 > 2$
$\rightarrow$ new element

$I = \{1, 5, 6\}$
$3 - r(0) = 3 - 0 = 3$
$3 \leq 3$
$\rightarrow$ no new element

Derived lattice

Steps of the proof:

- $D(\mathcal{L})$ is a lattice
- $\mathcal{L}^{op} \hookrightarrow D(\mathcal{L})$ in the "right" way
- $D(\mathcal{L})$ is a geometric lattice

Difficult part: semimodularity $r^*(x \vee y) + r^*(x \wedge y) \leq r^*(x) + r^*(y)$

Thank you for your attention.