Defining the q-analogue of a matroid

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q-Analogues

Finite set $\longrightarrow$ finite dimensional vectorspace over $\mathbb{F}_q$

Example

$$\binom{n}{k} = \text{number of sets of size } k \text{ contained in set of size } n$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \text{number of } k\text{-dim subspaces of } n\text{-dim vectorspace over } \mathbb{F}_q$$

$$= \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}$$

q-Analogues

finite set	finite space $\mathbb{F}_q^n$
element	1-dim subspace
size	dimension
n	$\frac{q^n-1}{q-1}$
intersection	intersection
union	sum
complement	?

From q-analogue to 'normal': let $q \to 1$.

Candidates for complement A^c of $A \subseteq \mathbb{F}_q^n$:

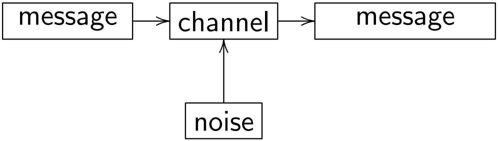
- All vectors outside A

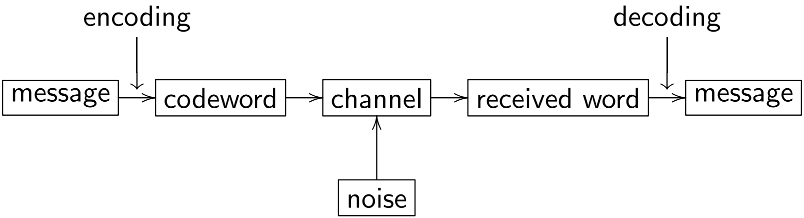
  But: not a space
- Orthogonal complement

  But: $A \cap A^\perp$ can be nontrivial
- Quotient space $\mathbb{F}_q^n / A$

  But: changes ambient space
- Subspace such that $A \oplus A^c = \mathbb{F}_q^n$

  But: not unique


message
channel
message
noise


encoding
decoding
message
codeword
channel
received word
message
noise

Linear codes

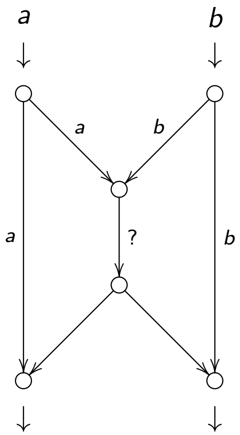
A linear code C is a subspace of $\mathbb{F}_q^n$.

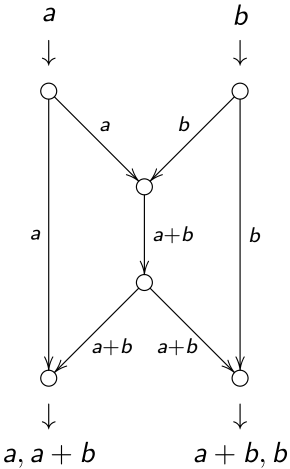
n length of the code

k dimension of the code

A $k \times n$ matrix whose rows span C is called a generator matrix.

Hamming distance: $d(\mathbf{x}, \mathbf{y}) = |\{i \in [n] : x_i \neq y_i\}|$.


a
b
a
b
a
?
b



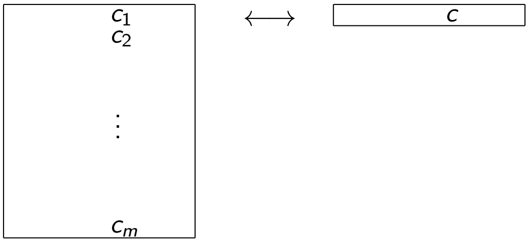
Idea: send (rows of) matrices instead of vectors

[Other idea: send (bases of) subspaces instead of matrices]

Rank metric codes

$\mathbb{F}_{q^m}/\mathbb{F}_q$ finite field extension with basis $\alpha_1, \ldots, \alpha_m$.

Write $c = c_1\alpha_1 + \cdots + c_m\alpha_m$.



$m(\mathbf{x}) \in \mathbb{F}_q^{m\times n}$ $\qquad\qquad$ $\mathbf{x} \in \mathbb{F}_{q^m}^n$

Rank metric code is subspace of $\mathbb{F}_{q^m}^n \to$ subspace of $\mathbb{F}_q^{m\times n}$.

Rank distance: $d(A, B) = \mathrm{rk}(A - B)$

C linear code

$\mathrm{supp}(\mathbf{c}) =$ coordinates of $\mathbf{c}$ that are non-zero

$\mathrm{wt}_H(\mathbf{c}) =$ size of support

Weight enumerator

$$W_C(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w$$

with $A_w =$ number of words of weight w.

C rank metric code

Rsupp$(\mathbf{c}) =$ row space of $m(\mathbf{c})$

$\mathrm{wt}_R(\mathbf{c}) =$ dimension of support

Rank weight enumerator

$$W_C^R(X, Y) = \sum_{w=0}^{n} A_w^R X^{n-w} Y^w$$

with $A_w^R =$ number of words of rank weight w.

J subset of $[n]$

$$C(J) = \{\mathbf{c} \in C : \mathrm{supp}(\mathbf{c}) \subseteq J^c\}$$

Lemma

$C(J)$ is a subspace of $\mathbb{F}_q^n$

$$\ell(J) = \dim_{\mathbb{F}_q} C(J)$$

Theorem

$\ell(J)$ gives a nice formula for the weight enumerator.

J subspace of $\mathbb{F}_q^n$

$$C(J) = \{\mathbf{c} \in C : \mathrm{Rsupp}(\mathbf{c}) \subseteq J^{\perp}\}$$

Lemma

$C(J)$ is a subspace of $\mathbb{F}_{q^m}^n$

$$\ell(J) = \dim_{\mathbb{F}_{q^m}} C(J)$$

Theorem

$\ell(J)$ gives a nice formula for the rank weight enumerator.

A linear code is a matroid, represented by a generator matrix.

Theorem

$\ell(J) = \dim C - r(J)$

Corollary

The Tutte polynomial of a matroid determines the (extended) weight enumerator of the corresponding code, via $\ell(J)$*.*

ULTIMATE GOAL: find a q-analogue of this.

Theorem

Let $r(J) = \dim C - \ell(J)$ *for a rank metric code* C*. Then* $r(J)$ *satisfies for all* $A, B \subseteq E$*:*

(r1) $0 \leq r(A) \leq \dim A$

(r2) *If* $A \subseteq B$ *then* $r(A) \leq r(B)$*.*

(r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$

q-Matroid: a pair (E, r) with

- E finite dimensional vector space;
- $r : \{\text{subspaces of } E\} \to \mathbb{N}_0$ a function, the *rank function*, that satisfies (r1),(r2),(r3).

Let $e \subseteq E$ a 1-dimension subspace.

Restriction (or: deletion of e)

Ground space: hyperplane $e^{\perp}$

Rank: $r_{M-e}(A) = r_M(A)$

Contraction of e

Ground space: quotient E/e, with a projection $\pi : E \to E/e$

Rank: $r_{M/e}(A) = r_M(B) - 1$,
with $B \subseteq E$ unique such that $e \subseteq B$ and $\pi(B) = A$

Duality

Ground space: E

Rank: $r_{M^*}(A) = r_M(A^\perp) + \dim A - r(M)$

$\Rightarrow$ Bases: orthogonal complements of bases of M

First deletion, then duality $=$ first duality, then contraction

Independent spaces: $\mathcal{I} = \{I \subseteq E : r(I) = \dim I\}$

Theorem

The independent spaces of a q-matroid satisfy:

(I1) $\mathbf{0} \in \mathcal{I}$

(I2) *If* $A \in \mathcal{I}$ *and* $B \subseteq A$ *then* $B \in \mathcal{I}$.

(I3) *If* $A, B \in \mathcal{I}$ *and* $\dim A > \dim B$ *then there is a 1-dimensional subspace* $a \subseteq A$, $a \not\subseteq B$ *such that* $B + a \in \mathcal{I}$.

Opposite is not true...

Example

Let $E = \mathbb{F}_2^4$ and $\mathcal{I} = \left\{ \left\langle \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right\rangle \text{ and all its subspaces} \right\}$.

$\mathcal{I}$ satisfies (I1),(I2),(I3), and r satisfies (r1),(r2). But:

$$A = \left\langle \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\rangle \quad B = \left\langle \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} \right\rangle$$

Then $r(A+B) + r(A \cap B) = 2 + 0 > 1 + 0 = r(A) + r(B)$!

Problem: embedding $\mathcal{I}$ in a bigger space does not give a q-matroid.

Solution: extra axiom for independence.

(I4) Let $x \notin \mathcal{I}$, $A \subseteq E$ and I a maximal independent space in A. Then $I + x$ is a maximal independent space in $A + x$.

Or: if $r(A) = r(A + x) = r(A + y)$, then $r(A) = r(A + x + y)$.

Theorem

Let $\mathcal{I}$ satisfy (I1)–(I4). Then $\mathcal{I}$ is the collection of independent spaces of a q-matroid.

What's next?

- More cryptomorphic descriptions (bases, circuits, flats, ...)
- Define q-analogue of Tutte polynomial
- Link with rank weight enumerator
- Rank metric codes that are not $\mathbb{F}_{q^m}$-linear
- Link with puncturing/shortening of rank metric codes?
- Link with other q-analogues?
- Do all q-matroids come from rank metric codes?

- Pick your favorite matroid theorem and find a q-analogue!