A q-analogue of perfect matroid designs

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Matroid: a pair (E, r) with

- E finite set;
- $r : 2^E \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \in E$:
  (r1) $0 \leq r(A) \leq |A|$
  (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  (r3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Examples:

- Set of vectors; rank = matrix rank
  In particular: columns of generator matrix of linear code
- Set of edges of a graph; rank = size of spanning tree

A subset $F \subseteq E$ is a flat if $r(F \cup \{x\}) > r(F)$ for any $x \notin F$.

The closure of a subset $A \subseteq E$ is the smallest flat that contains A.

Flats are equal to their closure: *closed sets*.

(If $r(A) = |A|$ the set is called *independent*.)

A matroid is also a pair $(E, \mathcal{F})$ with

- E finite set;
- $\mathcal{F} \subseteq 2^E$ a collection of subsets, the *flats*, with:
  - (F1) $E \in \mathcal{F}$
  - (F2) If $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$.
  - (F3) If $F \in \mathcal{F}$, then every $x \notin F$ is in a unique flat covering F.

We get all flats by taking intersections of rank $r(E) - 1$ flats.

A perfect matroid design is a matroid such that all flats of the same rank have the same size.

Example

- (Truncations of) projective spaces;
- (Truncations of) affine spaces;
- Steiner systems;
- Rank 4 PMDs coming from Moufang loops.

Theorem (Murty, Young & Edmonds, 1970)

The independent sets / circuits / flats of size j form a design.

q-analogue: finite set $\longrightarrow$ finite vector space over $\mathbb{F}_q$

Example

$$\binom{n}{k} = \text{number of sets of size } k \text{ contained in set of size } n$$

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \text{number of } k\text{-dim subspaces of } n\text{-dim vector space over } \mathbb{F}_q \\ &= \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i} \end{aligned}$$

Example

t-(v, k, λ) design: pair $(X, \mathcal{B})$ with

- X set with v elements (points)
- $\mathcal{B}$ family of subsets of X of size k (blocks)
- Every t-tuple of points is contained in exactly λ blocks

t-$(v, k, \lambda; q)$ subspace design: pair $(X, \mathcal{B})$ with

- X v-dim vectorspace over $\mathbb{F}_q$
- $\mathcal{B}$ family of k-dim subspaces of X (blocks)
- Every t-dim subspace is contained in exactly λ blocks

If $\lambda = 1$ we call the (subspace) design a *(q-)Steiner system*

finite set	finite space $\mathbb{F}_q^n$
element	1-dim subspace
size	dimension
n	$\frac{q^n-1}{q-1}$
intersection	intersection
union	sum
complement	(it depends)

From q-analogue to 'normal': let $q \to 1$.

q-Matroid: a pair (E, r) with

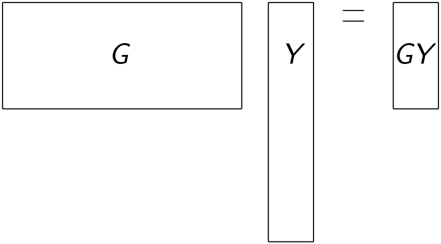
- E finite dimensional vector space;
- $r : \{\text{subspaces of } E\} \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \subseteq E$:
  - (r1) $0 \leq r(A) \leq \dim A$
  - (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  - (r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Theorem (J. & Pellikaan, 2016)

Every $\mathbb{F}_{q^m}$-linear rank metric code gives a q-matroid.

Proof.

Let $E = \mathbb{F}_q^n$ and G be a generator matrix of the code.
Let $A \subseteq E$ and Y a matrix whose columns span A.



Then $r(A) = \mathrm{rk}(GY)$ satisfies the axioms (r1),(r2),(r3). $\square$

A q-matroid is also a pair $(E, \mathcal{F})$ with

- E finite set;
- $\mathcal{F}$ a collection of subspaces, the *flats*, with:
  - (F1) $E \in \mathcal{F}$
  - (F2) If $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$.
  - (F3) If $F \in \mathcal{F}$, then every 1-dimensional subspace $x \not\subseteq F$ is in a unique flat covering F.

We get all flats by taking intersections of rank $r(E) - 1$ flats.

A q-PMD is a q-matroid such that all flats of the same rank have the same dimension.

Lemma

q-Steiner systems are q-PMDs, where the blocks are maximal proper flats and the rank function is

$$r(A) = \begin{cases} \dim A & \textit{if } \dim A \leq t \\ t & \textit{if } \dim A > t \textit{ and } A \textit{ is contained in a block} \\ t+1 & \textit{if } \dim A > t \textit{ and } A \textit{ is not contained in a block} \end{cases}$$

Fact: finding q-Steiner systems is hard. Maybe q-matroids help?

Conjecture (J. & Torielli, 2017)

All q-matroids come from rank metric codes.

That means: a q-matroid over $E = \mathbb{F}_q^n$ of rank k can be represented by a $k \times n$ matrix over a suitably large extension field $\mathbb{F}_{q^m}$.

Example

t-$(v, t, 1; q)$ subspace design: all t-spaces are blocks.

$$r(A) = \begin{cases} \dim A & \text{if } \dim A \leq t \\ t+1 & \text{if } \dim A > t \end{cases}$$

Uniform q-matroid of rank $t+1$ comes from an MRD code.

Example

1-$(v, k, 1; q)$ subspace design: spread.

$v = 6, k = 3, q = 2$ over $\mathbb{F}_8$:

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & \alpha^2 \end{pmatrix}$$

$v = 6, k = 2, q = 2$ over $\mathbb{F}_8$:

$$\begin{pmatrix} 1 & \alpha & 0 & 0 & 1 & \alpha \\ 0 & 0 & 1 & \alpha & \alpha^2 & \alpha^3 \end{pmatrix}$$

To do list:

- Fix details.
- Do q-PMDs give us subspace designs?
- Do other results on PMDs have a q-analogue? (Deza, 1992)
- Residual/derived design vs deletion/contraction in q-matroid.
- Relation between the representation matrix and the automorphisms of a design?
- How to decide if a matrix gives a q-PMD?
- Find a representation of the $\mathcal{S}_2(2,3,13)$ q-Steiner system.
- Wishful thinking: what about the q-analogue of the Fano plane ...?

Help is welcome!