# Codes and related combinatorial structures

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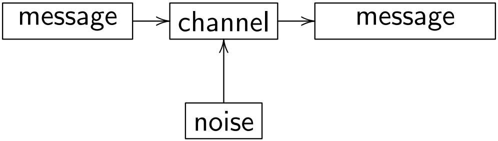
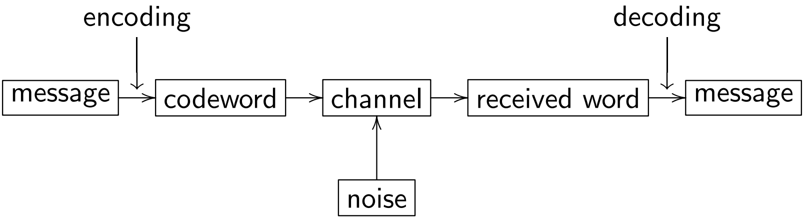

message
channel
message
noise



encoding
decoding
message
codeword
channel
received word
message
noise

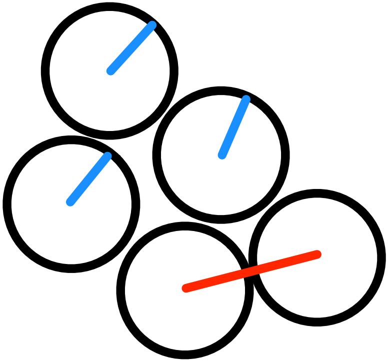
Alphabet $\mathcal{Q}$

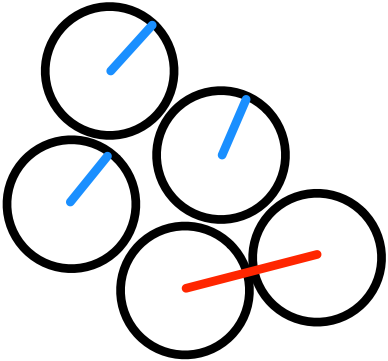
Length $n$

Hamming metric on $\mathcal{Q}^n$:

$$\begin{aligned} d(x,y) &= \text{number of positions in which vectors differ} \\ &= |\{i \in [n] : x_i \neq y_i\}| \end{aligned}$$

error-correcting code: $C \subseteq \mathcal{Q}^n$





$d$ minimum distance

$e$ error-correcting capacity

$$= \lfloor \frac{d-1}{2} \rfloor$$

A linear code $C$ is a subspace of $\mathbb{F}_q^n$.

- $n$ length of the code
- $k$ dimension of the code

A $k \times n$ matrix whose rows span $C$ is called a generator matrix.

## Running example

The $[7, 4]$ Hamming code over $\mathbb{F}_2$ has generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It has length $n = 7$, dimension $k = 4$ and minimum distance $d = 3$. It can correct 1 error.

Some research directions:

- Given two of $n, k, d$; optimize the third
- (Non)existence, constructions
- Efficient decoding
- Bounds on the parameters
- Links with other combinatorial objects

$C$ linear code

$$\mathrm{supp}(\mathbf{c}) = \text{ coordinates of } \mathbf{c} \text{ that are non-zero}$$
$$\mathrm{wt}_H(\mathbf{c}) = \text{ size of support}$$

Weight enumerator

$$W_C(X,Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w$$

with $A_w =$ number of words of weight $w$.

## Example

Hamming code generated by $G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

Some words and weights:

| $\mathbf{c}$ | $\mathrm{supp}(\mathbf{c})$ | $\mathrm{wt}_H(\mathbf{c})$ |
|---|---|---|
| $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | $\emptyset$ | 0 |
| $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$ | $\{1, 5, 6\}$ | 3 |
| $\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$ | $\{2, 3, 4, 7\}$ | 4 |

The weight enumerator is equal to

$$W_C(X, Y) = X^7 + 7X^4Y^3 + 7X^3Y^4 + Y^7.$$

$J$ subset of $[n]$

$$C(J) = \{\mathbf{c} \in C : \mathrm{supp}(\mathbf{c}) \subseteq J^c\}$$

Lemma

*$C(J)$ is a subspace of $\mathbb{F}_q^n$*

$$\ell(J) = \dim_{\mathbb{F}_q} C(J)$$

Theorem

*$\ell(J)$ gives a nice formula for the weight enumerator.*

Example

Hamming code generated by $G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

Some calculations of $C(J)$:

| $J$ | $J^c$ | $C(J)$ |
|---|---|---|
| $[n]$ | $\emptyset$ | $\mathbf{0}$ |
| size $= 5$ | size $= 2$ | $\mathbf{0}$ |
| $\{1, 5, 6\}$ | $\{2, 3, 4, 7\}$ | $\left\langle \begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{matrix} \right\rangle$ |
| $\{1\}$ | $\{2, 3, 4, 5, 6, 7\}$ | $\left\langle \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix} \right\rangle$ |

Matroid: a pair $(E, \mathcal{I})$ with

- $E$ finite set;
- $\mathcal{I} \subseteq 2^E$ family of subsets of $E$, the *independent sets*, with:
  - (I1) $\emptyset \in \mathcal{I}$
  - (I2) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
  - (I3) If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

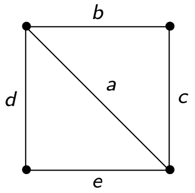
Examples:

- Set of vectors; independence = linear independence
- Set of edges of a graph; independence = cycle free

Example

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Example




But: most matroids don't come from a matrix or graph.

Matroids are everywhere: graphs, linear algebra, optimization, tropical geometry, hyperplane arrangements, topology, ...

Some research directions:

- Does a matroid come from a graph?
- Does a matroid come from a set of vectors? Over which field?
- How many matroids are there?
- Study concrete class of matroids
- Links with other combinatorial objects

A matroid is also a pair $(E, r)$ with

- $E$ finite set;
- $r : 2^E \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \in E$:
  (r1) $0 \leq r(A) \leq |A|$
  (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  (r3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

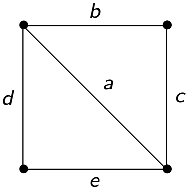
$r(A) =$ size of largest independent set contained in $A$

$\mathcal{I} = \{$subsets whose size is equal to their rank$\}$

Example

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Example



Fact: a linear code gives a matroid with

$E =$ index set for columns of generator matrix

$r(J) =$ dimension of subspace spanned by vectors indexed by $J$

Recall: $\ell(J) = \dim\{\mathbf{c} \in C : \mathrm{supp}(\mathbf{c}) \subseteq J^c\}$

Theorem

$r(J) = \dim C - \ell(J)$

Idea of proof: $0 \to C(J) \to C \to C_J \to 0$ is an exact sequence.

# Comercial break

Source: Ministry of Defence

# $q$-Analogues

Finite set $\longrightarrow$ finite dimensional vectorspace over $\mathbb{F}_q$

Example

$$\binom{n}{k} \quad = \quad \text{number of sets of size } k \text{ contained in set of size } n$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \quad = \quad \text{number of } k\text{-dim subspaces of } n\text{-dim vectorspace over } \mathbb{F}_q$$

# $q$-Analogues

Example

$t$-$(v, k, \lambda)$ design: pair $(X, \mathcal{B})$ with

- $X$ set with $v$ elements (points)
- $\mathcal{B}$ family of subsets of $X$ of size $k$ (blocks)
- Every $t$-tuple of points is contained in exactly $\lambda$ blocks

$t$-$(v, k, \lambda; q)$ $q$-design: pair $(X, \mathcal{B})$ with

- $X$ $v$-dim vectorspace over $\mathbb{F}_q$
- $\mathcal{B}$ family of $k$-dim subspaces of $X$ (blocks)
- Every $t$-dim subspace is contained in exactly $\lambda$ blocks

# $q$-Analogues

| finite set | finite space $\mathbb{F}_q^n$ |
|---|---|
| element | 1-dim subspace |
| size | dimension |
| $n$ | $\frac{q^n-1}{q-1}$ |
| intersection | intersection |
| union | sum |
| complement | ? |

From $q$-analogue to ‘normal’: let $q \to 1$.

Candidates for complement $A^c$ of $A \subseteq \mathbb{F}_q^n$:

- All vectors outside $A$

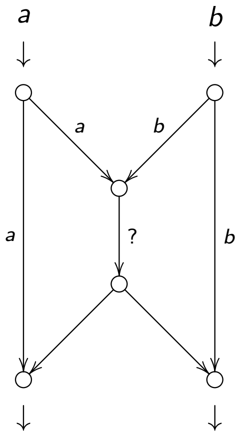
  But: not a space
- Orthogonal complement

  But: $A \cap A^\perp$ can be nontrivial
- Quotient space $\mathbb{F}_q^n / A$

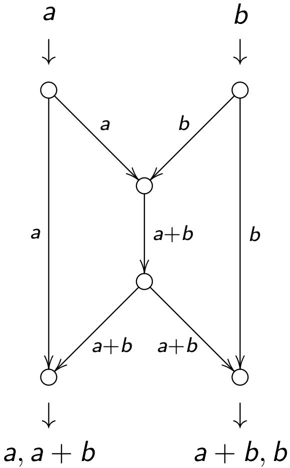
  But: changes ambient space
- Subspace such that $A \oplus A^c = \mathbb{F}_q^n$

  But: not unique

# Network coding



# Network coding




Idea: send (rows of) matrices instead of vectors

Better idea: send (bases of) subspaces instead of matrices

Codewords are vectors:

'Ordinary' error-correcting codes

Codewords are matrices:

Rank metric codes

$q$-analogue of 'ordinary' codes

Codewords are subspaces:

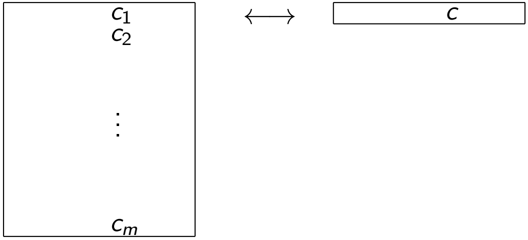
Subspace codes

Constant dimension, constant weight: $q$-design

# Rank metric codes

$\mathbb{F}_{q^m}/\mathbb{F}_q$ finite field extension with basis $\alpha_1, \ldots, \alpha_m$.

Write $c = c_1\alpha_1 + \cdots + c_m\alpha_m$.




$m(\mathbf{x}) \in \mathbb{F}_q^{m\times n}$ $\qquad$ $\mathbf{x} \in \mathbb{F}_{q^m}^n$

Rank metric code is subspace of $\mathbb{F}_{q^m}^n \to$ subspace of $\mathbb{F}_q^{m\times n}$.

## Running example

Let $\mathbb{F}_8/\mathbb{F}_2$ with basis $(1, \alpha, \alpha^2)$, where $\alpha^3 = \alpha + 1$.
Let $C$ be the code generated by

$$G = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \end{pmatrix}.$$

Some words and matrices:

$$\begin{array}{cccc} \mathbf{c} & (0,0,0,0) & (1,\alpha,0,0) & (1,\alpha^2,\alpha^5,0) \\ m(\mathbf{c}) & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$C$ linear code

$$\mathrm{supp}(\mathbf{c}) = \text{ coordinates of } \mathbf{c} \text{ that are non-zero}$$

$$\mathrm{wt}_H(\mathbf{c}) = \text{ size of support}$$

Weight enumerator

$$W_C(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w$$

with $A_w$ = number of words of weight $w$.

$C$ rank metric code

$\mathrm{Rsupp}(\mathbf{c}) =$ row space of $m(\mathbf{c})$

$\mathrm{wt}_R(\mathbf{c}) =$ dimension of support

Rank weight enumerator

$$W_C^R(X, Y) = \sum_{w=0}^{n} A_w^R X^{n-w} Y^w$$

with $A_w^R =$ number of words of rank weight $w$.

Example

$C$ in $\mathbb{F}_8/\mathbb{F}_2$ generated by $G = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \end{pmatrix}$.

Some words and weights:

$$\begin{array}{llll}
\mathbf{c} & (0,0,0,0) & (1,\alpha,0,0) & (1,\alpha^2,\alpha^5,0) \\
m(\mathbf{c}) & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\
\mathrm{Rsupp}(\mathbf{c}) & \mathbf{0} & \left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\rangle & \left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \right\rangle \\
\mathrm{wt}_R(\mathbf{c}) & 0 & 2 & 3
\end{array}$$

There are no words of weight 1 or weight 4.

$J$ subset of $[n]$

$$C(J) = \{\mathbf{c} \in C : \mathrm{supp}(\mathbf{c}) \subseteq J^c\}$$

Lemma

*$C(J)$ is a subspace of $\mathbb{F}_q^n$*

$$\ell(J) = \dim_{\mathbb{F}_q} C(J)$$

Theorem

*$\ell(J)$ gives a nice formula for the weight enumerator.*

$J$ subspace of $\mathbb{F}_q^n$

$$C(J) = \{\mathbf{c} \in C : \mathrm{Rsupp}(\mathbf{c}) \subseteq J^{\perp}\}$$

Lemma
*$C(J)$ is a subspace of $\mathbb{F}_{q^m}^n$*

$$\ell(J) = \dim_{\mathbb{F}_{q^m}} C(J)$$

Theorem
*$\ell(J)$ gives a nice formula for the rank weight enumerator.*

Example

$C$ in $\mathbb{F}_8/\mathbb{F}_2$ generated by $G = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \end{pmatrix}$.

Some calculations of $C(J)$:

| $J$ | $J^\perp$ | $C(J)$ |
|---|---|---|
| $\mathbb{F}_2^4$ | $\mathbf{0}$ | $\mathbf{0}$ |
| $\dim = 3$ | $\dim = 1$ | $\mathbf{0}$ |
| $\left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\rangle$ | $\left\langle \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\rangle$ | $\mathbf{0}$ |
| $\left\langle \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\rangle$ | $\left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\rangle$ | $\left\langle \begin{matrix} 1 & \alpha & 0 & 0 \end{matrix} \right\rangle$ |

If $\dim J = 2$ and $J^\perp \not\subseteq \left\langle \begin{matrix} 0 & 0 & 0 & 1 \end{matrix} \right\rangle^\perp$ then $C(J) = \mathbf{0}$.

Recall: for linear codes and their associated matroids, we have proven $r(J) = \dim C - \ell(J)$.

Theorem

*Let $r(J) = \dim C - \ell(J)$ for a rank metric code $C$. Then $r(J)$ satisfies:*

(r1) $0 \leq r(A) \leq \dim A$

(r2) *If $A \subseteq B$ then $r(A) \leq r(B)$.*

(r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ *(semimodular)*

This leads to the definition of the rank function of a $q$-matroid.

$q$-Matroid: a pair $(E, r)$ with

- $E$ finite dimensional vector space;
- $r : \{\text{subspaces of } E\} \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \subseteq E$:
  - (r1) $0 \leq r(A) \leq \dim A$
  - (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  - (r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Example

$C$ in $\mathbb{F}_8/\mathbb{F}_2$ generated by $G = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 \end{pmatrix}$.

Some calculations of $r(J)$:

| $J$ | $C(J)$ | $r(J)$ |
|---|---|---|
| $\mathbb{F}_2^4$ | $\mathbf{0}$ | 2 |
| $\dim = 3$ | $\mathbf{0}$ | 2 |
| $\left\langle \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\rangle$ | $\mathbf{0}$ | 2 |
| $\left\langle \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\rangle$ | $\langle \begin{matrix} 1 & \alpha & 0 & 0 \end{matrix} \rangle$ | 1 |

If $\dim J = 2$ and $J^\perp \not\subseteq \langle \begin{matrix} 0 & 0 & 0 & 1 \end{matrix} \rangle^\perp$ then $r(J) = 2$.

These are exactly the bases of the $q$-matroid.

Theorem

*A q-matroid is a pair $(E, \mathcal{I})$ with*

- *$E$ finite dimensional vector space;*
- *$\mathcal{I}$ family of subspaces of $E$, the independent spaces, with:*
  - (I1) $\mathbf{0} \in \mathcal{I}$.
  - (I2) *If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*
  - (I3) *If $I, J \in \mathcal{I}$ with $\dim I < \dim J$, then there is some 1-dimensional subspace $x \subseteq J$, $x \not\subseteq I$ with $I + x \in \mathcal{I}$.*
  - (I4) *Let $A, B \subseteq E$ and let $I, J$ be maximal independent subspaces of $A$ and $B$, respectively. Then there is a maximal independent subspace of $A + B$ that is contained in $I + J$.*

Note the extra axiom!

Other things that need a $q$-analogue:

- More axiom sets for matroids
- Relation between weight enumerator and Tutte polynomial
- Perfect matroid designs
- Graphs (??)

- How about rank metric cods that are not $\mathbb{F}_{q^m}$-linear?