

The q -analogue of a matroid

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q -Analogues

Finite set \longrightarrow finite dimensional vectorspace over \mathbb{F}_q

Example

$\binom{n}{k}$ = number of sets of size k contained in set of size n

$\left[\begin{matrix} n \\ k \end{matrix} \right]_q$ = number of k -dim subspaces of n -dim vectorspace over \mathbb{F}_q

q -Analogues

finite set	finite space \mathbb{F}_q^n
element	1-dim subspace
size	dimension
n	$\frac{q^n - 1}{q - 1}$
intersection	intersection
union	sum
complement	it depends

From q -analogue to 'normal': let $q \rightarrow 1$.

Candidates for complement A^c of $A \subseteq \mathbb{F}_q^n$:

- ▶ All vectors outside A
But: not a space
- ▶ Orthogonal complement
But: $A \cap A^\perp$ can be nontrivial
- ▶ Quotient space \mathbb{F}_q^n/A
But: changes ambient space
- ▶ Subspace such that $A \oplus A^c = \mathbb{F}_q^n$
But: not unique

q-Matroid: a pair (E, r) with

- ▶ E finite dimensional vector space;
- ▶ $r : \{\text{subspaces of } E\} \rightarrow \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \subseteq E$:
 - (r1) $0 \leq r(A) \leq \dim A$
 - (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
 - (r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Theorem (J. & Pellikaan, 2016)

Every \mathbb{F}_{q^m} -linear rank metric code gives a q -matroid.

Proof.

Let $E = \mathbb{F}_q^n$ and G be a generator matrix of the code.

Let $A \subseteq E$ and Y a matrix whose columns span A .

$$\boxed{G} \quad \boxed{Y} = \boxed{GY}$$

Then $r(A) = \text{rk}(GY)$ satisfies the axioms $(r1),(r2),(r3)$. □

Lemma

Matrix representation is equivalent under

- ▶ *row operations over \mathbb{F}_{q^m} ;*
- ▶ *column operations over \mathbb{F}_q .*

We call a q -matroid that comes from a code *representable*.

Conjecture (J. & Torielli, 2017)

All q -matroids are representable.

That means: a q -matroid over $E = \mathbb{F}_q^n$ of rank k can be represented by a $k \times n$ matrix over a suitably large extension field \mathbb{F}_{q^m} .

Motivating evidence:

- ▶ uniform q -matroids are representable;
- ▶ the matrix has entries in an extension field.

A q -matroid could also be a pair (E, \mathcal{I}) with

- ▶ E finite dimensional vector space;
- ▶ \mathcal{I} family of subspaces of E , the *independent spaces*, with:
 - (I1) $\mathbf{0} \in \mathcal{I}$.
 - (I2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.
 - (I3) If $I, J \in \mathcal{I}$ with $\dim I < \dim J$, then there is some 1-dimensional subspace $x \subseteq J$, $x \not\subseteq I$ with $I + x \in \mathcal{I}$.

$r(A)$ = dimension of largest independent space contained in A

$\mathcal{I} = \{\text{subspaces whose dimension is equal to their rank}\}$

Example

Let $E = \mathbb{F}_2^4$ and $\mathcal{I} = \left\{ \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right\rangle \text{ and all its subspaces} \right\}$.

\mathcal{I} satisfies (I1),(I2),(I3), and r satisfies (r1),(r2). But:

$$A = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \quad B = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

Then $r(A + B) + r(A \cap B) = 2 + 1 > 1 + 1 = r(A) + r(B) !$

Problem: $(r_1), (r_2), (r_3) \Rightarrow (I_1), (I_2), (I_3)$; but not \Leftarrow .

Solution: find an extra axiom (I_4) for \mathcal{I}

Lemma

Loops come in subspaces.

Corollary

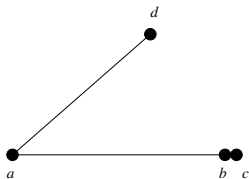
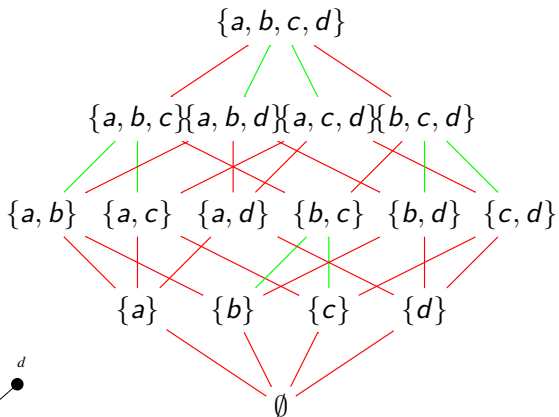
If an axiom set is invariant under embedding E in a bigger space, it can not be a full axiom set for \mathcal{I} .

Theorem

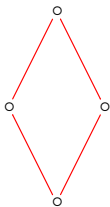
A q -matroid is a pair (E, \mathcal{I}) with

- ▶ E finite dimensional vector space;
- ▶ \mathcal{I} family of subspaces of E , the independent spaces, with:
 - (I1) $\mathcal{I} \neq \emptyset$.
 - (I2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.
 - (I3) If $I, J \in \mathcal{I}$ with $\dim I < \dim J$, then there is some 1-dimensional subspace $x \subseteq J$, $x \not\subseteq I$ with $I + x \in \mathcal{I}$.
 - (I4) Let $A, B \subseteq E$ and let I, J be maximal independent subspaces of A and B , respectively. Then there is a maximal independent subspace of $A + B$ that is contained in $I + J$.

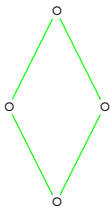
Example



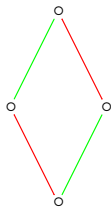
Matroid \iff only the following diamonds:



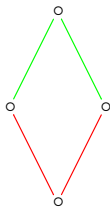
one



zero



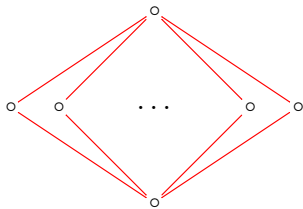
mixed



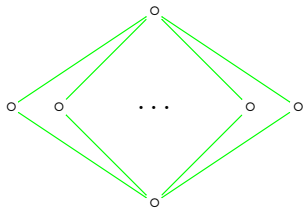
prime

q -analogue: change Boolean lattice to subspace lattice
(or another complemented modular lattice)

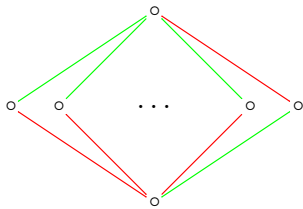
q -Matroid \iff only the following “diamonds”:



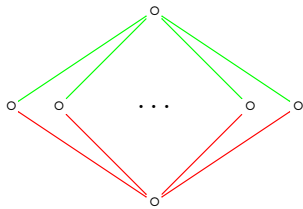
one



zero

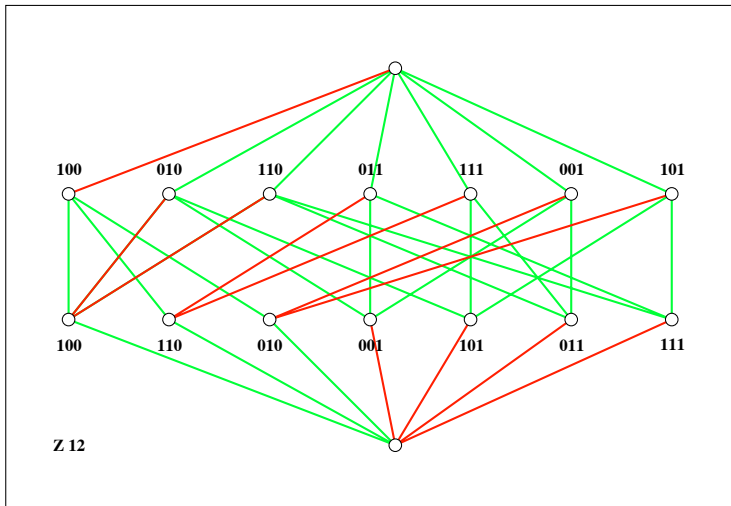


mixed

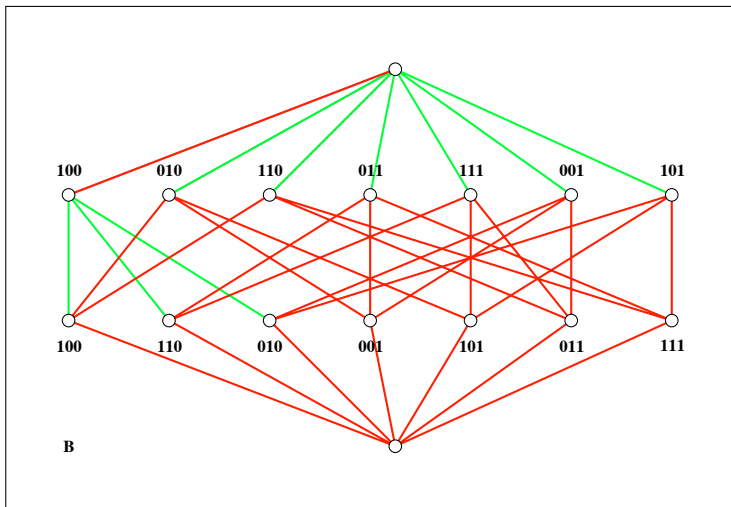


prime

Example



Example



Open questions

- ▶ Axiom system for circuits?
- ▶ Deletion/contraction for the q -Tutte polynomial?
- ▶ Internal and external activity of a basis?

Theorem

Let $M = (E, \mathcal{I})$ be a q -matroid and \mathcal{C} its family of circuits. Then \mathcal{C} satisfies:

(C1) $\mathbf{0} \notin \mathcal{C}$

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ distinct and $x \subseteq C_1 \cap C_2$ a 1-dimensional subspace, then there is a $C_3 \subseteq C_1 + C_2$ with $x \not\subseteq C_3$ so that $C_3 \in \mathcal{C}$.

... but we need another axiom.

Rank generating polynomial:

$$R(x, y) = \sum_{A \subseteq E} x^{r(M)-r(A)} y^{\dim(A)-r(A)}$$

Tutte polynomial:

classical: $x \rightarrow x - 1, y \rightarrow y - 1$

q: something similar but with powers of q ??

Duality

Let $r^*(A) = \dim A - r(M) + r(A^\perp)$.

Theorem

$M^* = (E, r^*)$ is a q -matroid, i.e., r^* satisfies (r1),(r2),(r3).

Lemma

$\mathcal{B}(M^*)$ are the orthogonal complements of bases of M .

Restriction

Let H a hyperplane in E and let $r_{M|_H}(A) = r_M(A)$.

Theorem

$M|_H = (H, r_{M|_H})$ is a q -matroid, i.e., $r_{M|_H}$ satisfies (r1),(r2),(r3).

Lemma

$\mathcal{I}(M|_H)$ are the independent spaces of M that are contained in H .

Contraction

Let e 1-dim subspace of E ,

$\pi : E \rightarrow E/e$ projection,

A in E/e and B in E such that $e \subseteq B$ and $\pi(B) = A$.

Let $r_{M/e}(A) = r_M(B) - r_M(e)$.

Theorem

$M/e = (E/e, r_{M/e})$ is a q -matroid, i.e., $r_{M/e}$ satisfies $(r1), (r2), (r3)$.

Lemma

$\mathcal{I}(M/e)$ are the independent spaces of M that contain e , projected to E/e .

Restriction, contraction, and duality

Theorem

Restriction and contraction are dual operations:

$$(M/e)^* = M^*|_{e^\perp} \text{ and } (M|_{e^\perp})^* = M^*/e.$$

$$\begin{aligned}
 R_M(X, Y) &= \sum_{\substack{A \subseteq E \\ e \notin A}} X^{r(E)-r(A)} Y^{|A|-r(A)} + \sum_{\substack{A \subseteq E \\ e \in A}} X^{r(E)-r(A)} Y^{|A|-r(A)} \\
 &= R_{M-e}(X, Y) + R_{M/e}(X, Y)
 \end{aligned}$$

e element of finite set E

$$\{\text{subsets containing } e\} \cup \{\text{subsets of } e^c\} = 2^E$$

e 1-dimensional subspace of \mathbb{F}_q^n

$$\{\text{subspaces containing } e\} \cup \{\text{subspaces of } e^\perp\} \neq \{\text{subspaces of } \mathbb{F}_q^n\}$$

Original Tutte polynomial:

$$T(x, y) = \sum_{B \in \mathcal{B}} x^{i(B)} y^{e(B)}$$

with $i(B)$ and $e(B)$ the number of **internally** and **externally active** elements of B .

$e \in E \setminus B$ is externally active if it is the smallest element of the unique circuit contained in $B \cup \{e\}$. (internally active: dual)

Internal/external activity uses ordering on elements of the matroid.

Ordering on 1-dimensional subspaces ??

Element e does not need to be unique!

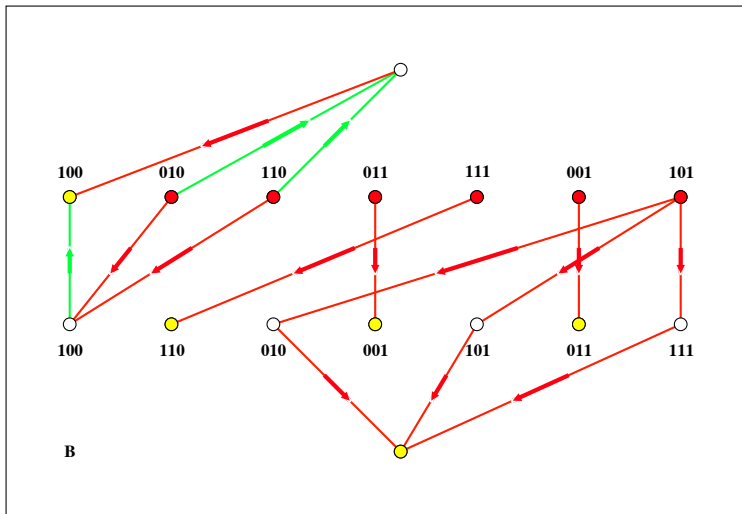
Internal/external activity induces partition of lattice in prime-free minors; that gives the Tutte polynomial.

classical: every part contains a basis

q : several bases per part, what is the right partition?

So the q -Tutte polynomial is a sum over parts of the partition: exponents of x and y depend on rank/nullity of the parts.

Example



$$T(x, y) = x^2 + xy + 3x$$

Open questions

- ▶ Axiom system for circuits?
- ▶ Deletion/contraction for the q -Tutte polynomial?
- ▶ Internal and external activity of a basis?

Or find a q -analogue of your favourite matroid result!