The combinatorial derived matroid

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Online graph & matroid seminar
February 16, 2023

What to tell about the (combinatorial) derived matroid?

- Motivation: existing definitions
- New definition
- Nice properties of the new definition
- Open questions

Warning: there will be *a lot* of handwaving.

Problem (Crapo and Rota, 1970's)

Given a matroid M, make a matroid with as ground set the set of (co)circuits of M and some 'natural' definition of dependence between them. In other words: study dependencies between dependencies.

Motivation: derived code (J., Pellikaan; 2014)

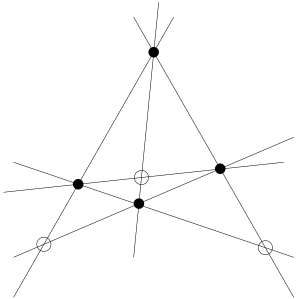
- Start with $[n, k]$ code.
- Consider the projective system $\mathcal{P}_C$.
- Look at all hyperplanes spanned by $k-1$ points of $\mathcal{P}_C$. (Ignore $k-1$ points that span spaces of lower dimension.)
- Remove (multiple) copies of hyperplanes.
- These hyperplanes form an arrangement $\mathcal{A}$.
- The *derived code* $D(C)$ is the code such that $\mathcal{A} = \mathcal{A}_{D(C)}$.

Motivation: derived code (J., Pellikaan; 2014)

Example

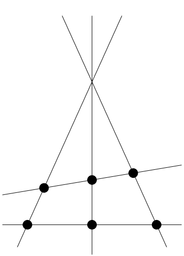
$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

C $\qquad\qquad$ $D(C)$



Motivation: derived code (J., Pellikaan; 2014)

Example



The derived code is not a matroid invariant. Can we make it one?

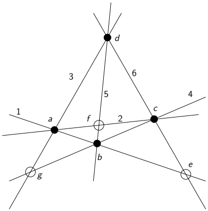
Motivation: derived code (J., Pellikaan; 2014)

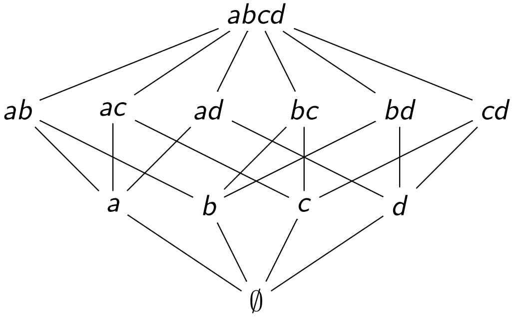
From a lattice point of view, this is how we make a derived code:

- Start with $[n, k]$ code.
- Consider the geometric lattice $\mathcal{L}(C)$.
- Turn it upside down: $\mathcal{L}^{opp}$.
- Add extra elements above atom level such that
  - $\mathcal{L}^{opp} \hookrightarrow \mathcal{L}(D(C))$ in the "right" way;
  - $\mathcal{L}(D(C))$ is a geometric lattice.

Motivation: derived code (J., Pellikaan; 2014)

Example





Motivation: derived code (J., Pellikaan; 2014)

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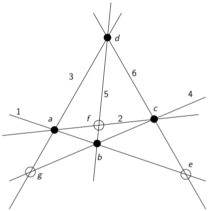
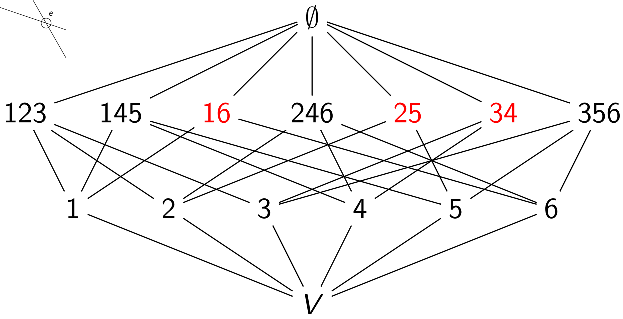




Motivation: adjoint

This geometric lattice-flipping-and-adding construction already has a name: *adjoint*.

Good news: every derived code $D(C)$ gives an adjoint of $M(C)$.

Bad news: not every geometric lattice has an adjoint (e.g., Vámos matroid) and if it exists, it is not unique.

Motivation: derived matroid (Oxley, Wang; 2018)

Longyear (1980) and Oxley, Wang (2018), translated:

- Start with $[n, k]$ code.
- Support of words in $C^\perp \leftrightarrow$ dependent sets of $M(C)$.
- $C^\perp$ is generated by words of minimal support, i.e., circuits vectors
- The *derived matroid* $\delta_{OW} M(C)$ is represented by these circuit vectors, taken as column vectors.

Motivation: derived matroid (Oxley, Wang; 2018)

Good news: up to duality, this is the same construction as the derived code $D(C)$.

Bad news: construction depends on representation, so not unique and not for arbitrarily matroids.

Motivation: derived matroid (Oxley, Wang; 2018)

Example

x	x	x	x				
x			x			x	x
	x	x				x	x

If there are more circuits vectors in a set then the dimension of $C^{\perp}$ restricted to the union of the supports of the circuits vectors, then the set is dependent in $\delta M(C)$.

In other words: $A \subseteq \mathcal{C}$ is dependent if $n(\cup_{C \in A} C) < |A|$.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

The idea behind our definition of the combinatorial derived matroid:

- Determine the necessarily dependent sets via nullity condition.
- Translate the dependence axioms into operations.
- Close the family of necessarily dependent sets under the dependence operations.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition
A matroid M is a pair $(E, \mathcal{D})$ where E is a finite set and $\mathcal{D}$ a family of subsetsof E, called the *dependent sets*, satisfying

(D1) $\emptyset \notin \mathcal{D}$;

(D2) if $D \in \mathcal{D}$ and $D \subseteq D'$ then $D' \in \mathcal{D}$;

(D3) if $D_1, D_2 \in \mathcal{D}$ and $D_1 \cap D_2 \notin \mathcal{D}$, then $(D_1 \cup D_2) \backslash \{e\} \in \mathcal{D}$ for all $e \in D_1 \cap D_2$.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition
Let $\mathcal{C}$ be the set of circuits of some matroid, and let $\mathcal{A} \subseteq \mathcal{C}$. Then we define the operations ϵ and $\uparrow$ on $\mathcal{A}$ as follows:

$$\epsilon(\mathcal{A}) = \mathcal{A} \cup \{(A_1 \cup A_2) \setminus \{C\} : A_1, A_2 \in \mathcal{A}, A_1 \cap A_2 \notin \mathcal{A}, C \in A_1 \cap A_2\},$$

$$\uparrow\mathcal{A} = \{A \subseteq \mathcal{C} : \exists A' \in \mathcal{A} : A' \subseteq A\}.$$

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition

Let M be a matroid, and $\mathcal{C} = \mathcal{C}(M)$ its collection of circuits. Define the collection

$$\mathcal{A}_0 := \{A \subseteq \mathcal{C} : |A| > n(\cup_{C \in A} C)\}.$$

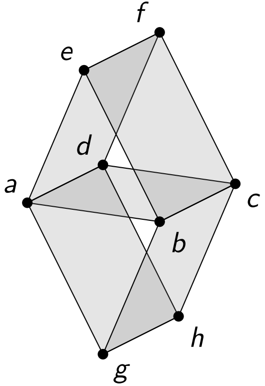
Inductively, we let $\mathcal{A}_{i+1} = \uparrow\epsilon(\mathcal{A}_i)$ for $i \geq 1$, and

$$\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_i.$$

Then $\delta M := (\mathcal{C}, \mathcal{A})$ is the *combinatorial derived matroid*.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Example (Vámos matroid)



δM has 41 points.

$A_1 = \{abcd, adef, bcef\}$
$A_2 = \{abcd, adgh, bcgh\}$
Both sets are in $\mathcal{A}_0$.

$A_3 = (A_1 \cup A_2) \setminus \{abcd\}$
This set is in $\mathcal{A}_1$ but not in $\mathcal{A}_0$.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Nice properties of the combinatorial derived matroid:

- We can define δM also by its circuits: alternate ϵ and taking inclusion-minimal sets.
- δM is connected if and only if M is connected.
- The rank of δM is at most $|E| - r(M)$.

Open questions

- Is the rank of δM equal to $|E| - r(M)$?
- Can we classify the independent sets / bases / dependent sets /circuits / flats / ... of δM without going to the whole iterative construction?
- Is there a precise way (e.g., with respect to the weak order) to say the combinatorial derived matroid is the 'most general' derived matroid?
- If M^* has an adjoint, is δM is isomorphic to an adjoint of M^*?

Thank you for your attention!