

Combinatorial derived matroids

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Combinatorial Optimization seminar TU/e

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What to tell about the (combinatorial) derived matroid?

- ▶ What is a matroid?
- ▶ What is the problem?
- ▶ Motivation: existing definitions of derived matroids
- ▶ New definition and nice properties
- ▶ Open questions

Warning: there will be *a lot* of handwaving.

What is a matroid?

Definition

A matroid M is a pair (E, \mathcal{D}) where E is a finite set and \mathcal{D} a family of subsets of E , called the *dependent sets*, satisfying

(D1) $\emptyset \notin \mathcal{D}$;

(D2) if $D \in \mathcal{D}$ and $D \subseteq D'$ then $D' \in \mathcal{D}$;

(D3) if $D_1, D_2 \in \mathcal{D}$ and $D_1 \cap D_2 \notin \mathcal{D}$, then $(D_1 \cup D_2) \setminus \{e\} \in \mathcal{D}$ for all $e \in D_1 \cap D_2$.

What is a matroid?

Example

Take a graph. Let E be the set of edges. Call a subset of edges dependent if it contains a cycle.

Example

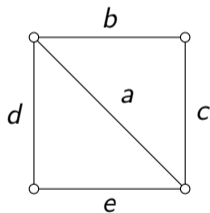
Take a matrix. Let E be the set of columns. Call a subset of columns dependent if they are linearly dependent.

What is a matroid?

Theorem

A graph and its incidence matrix over \mathbb{F}_2 give the same matroid.

Example



$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

But: not all matroids come from a graph or a matrix!

What is a matroid?

Some other useful matroid terms:

- ▶ Independent set: set that is not dependent
- ▶ Rank of a set: size of largest independent set contained in it
- ▶ Circuit: minimal dependent set
- ▶ Hyperplane: maximal non full-rank set
- ▶ Circuits and hyperplanes are dual/complementary notions

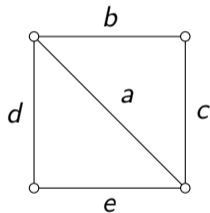
What is the problem?

Problem (Crapo and Rota, 1970's)

Given a matroid M , make a matroid with as ground set the set of (co)circuits of M and some 'natural' definition of dependence between them. In other words: study dependencies between dependencies.

What is the problem?

Example

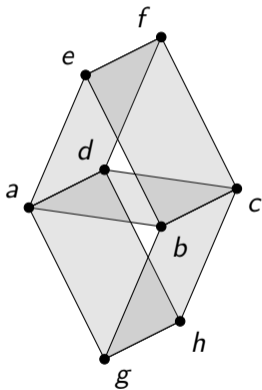


Circuits: abc , ade and $bcde$.

Intuition: the set of circuits $\{abc, ade, bcde\}$ should be dependent in the derived matroid.

What is the problem?

Example (Vámos matroid)



Circuits: the five “planes”, plus all 5-sets not containing them.

$$A_1 = \{abcd, adef, bcef\}$$

$$A_2 = \{abcd, adgh, bcgh\}$$

Intuition: these sets of circuits should be dependent in the derived matroid.

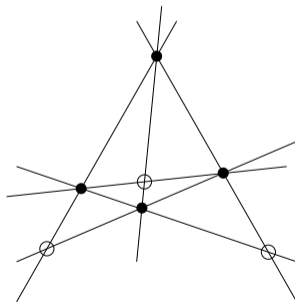
Motivation: derived code (J., Pellikaan; 2014)

- ▶ Start with (the rowspace of) a $k \times n$ matrix.
- ▶ Consider the columns as (projective) points.
- ▶ Look at all hyperplanes spanned by $k - 1$ points.
(Ignore $k - 1$ points that span spaces of lower dimension.)
- ▶ Remove (multiple) copies of hyperplanes.
- ▶ Make a matrix with as columns the vectors orthogonal to these hyperplanes.
- ▶ The *derived code* is the rowspace of this matrix.

Motivation: derived code (J., Pellikaan; 2014)

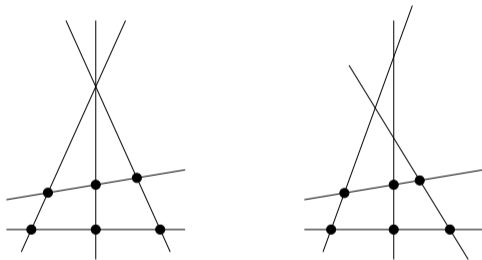
Example

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$



Motivation: derived code (J., Pellikaan; 2014)

Example



The derived code is not a matroid invariant. Can we make it one?

Motivation: derived matroid (Oxley, Wang; 2018)

Longyear (1980) and Oxley, Wang (2018), translated:

- ▶ Start with (the rowspace C of) a $k \times n$ matrix.
- ▶ Support of vectors in $C^\perp \leftrightarrow$ dependent sets of $M(C)$.
- ▶ C^\perp is generated by words of minimal support, i.e., circuit vectors.
- ▶ The *derived matroid* $\delta_{OW}M(C)$ is represented by these circuit vectors, taken as column vectors.

Motivation: derived matroid (Oxley, Wang; 2018)

Good news: up to duality, this is the same construction as the derived code.

Bad news: construction depends on choice of matrix for your matroid, so not unique and not for arbitrarily matroids.

Motivation: derived matroid (Oxley, Wang; 2018)

Example

x	x	x	x				
x			x			x	x
	x	x				x	x

If there are more circuits vectors in a set then the dimension of C^\perp restricted to the union of the supports of the circuits vectors, then the set is dependent in $\delta M(C)$.

In other words: $A \subseteq \mathcal{C}$ is dependent if $n(\cup_{C \in A} C) < |A|$.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

The idea behind our definition of the combinatorial derived matroid:

- ▶ Determine the necessarily dependent sets via nullity condition.
- ▶ Translate the dependence axioms into operations.
- ▶ Close the family of necessarily dependent sets under the dependence operations.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition

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New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition

Let \mathcal{C} be the set of circuits of some matroid, and let \mathcal{A} be a family of subsets of \mathcal{C} . Then we define the operations ϵ and \uparrow on \mathcal{A} as follows:

$$\epsilon(\mathcal{A}) = \mathcal{A} \cup \{(A_1 \cup A_2) \setminus \{C\} : A_1, A_2 \in \mathcal{A}, A_1 \cap A_2 \notin \mathcal{A}, C \in A_1 \cap A_2\},$$

$$\uparrow\mathcal{A} = \{A \subseteq \mathcal{C} : \exists A' \in \mathcal{A} : A' \subseteq A\}.$$

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Definition

Let M be a matroid, and \mathcal{C} its collection of circuits. Define the collection

$$\mathcal{A}_0 := \{A \subseteq \mathcal{C} : |A| > n(\cup_{C \in A} C)\}.$$

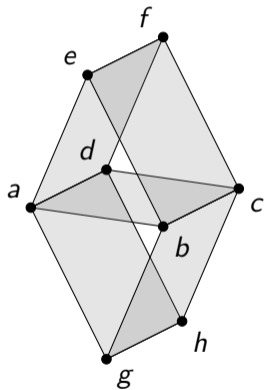
Inductively, we let $\mathcal{A}_{i+1} = \uparrow\epsilon(\mathcal{A}_i)$ for $i \geq 1$, and

$$\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_i.$$

Then $\delta M := (\mathcal{C}, \mathcal{A})$ is the *combinatorial derived matroid*.

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Example (Vámos matroid)



δM has 41 points.

$$A_1 = \{abcd, adef, bcef\}$$

$$A_2 = \{abcd, adgh, bcgh\}$$

Both sets are in \mathcal{A}_0 .

$$A_3 = (A_1 \cup A_2) \setminus \{abcd\}$$

This set is in \mathcal{A}_1 but not in \mathcal{A}_0 .

New definition (Freij-Hollanti, J., Kuznetsova; 2023)

Nice properties of the combinatorial derived matroid:

- ▶ We can define δM also by its circuits: alternate ϵ and taking inclusion-minimal sets.
- ▶ δM is connected if and only if M is connected.
- ▶ The rank of δM is at most $|E| - r(M)$.

Open questions

- ▶ Is the rank of δM equal to $|E| - r(M)$?
- ▶ Can we classify the independent sets / dependent sets / circuits / hyperplanes / ... of δM without going through the whole iterative construction?
- ▶ Is there a precise way (e.g., with respect to the weak order) to say the combinatorial derived matroid is the 'most general' derived matroid?
- ▶ If M^* has an adjoint, is δM isomorphic to an adjoint of M^* ?
- ▶ ...