A q-analogue of Δ-matroids

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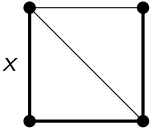
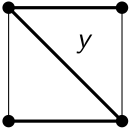
Spanning trees in graphs: exchange property

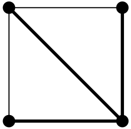
$B_1 =$



$B_2 =$



$B_1 - \{x\} \cup \{y\} =$



Matroid

Matroid: a pair $(E, \mathcal{B})$ with

- E finite set;
- $\mathcal{B}$ a family of subsets of E, the *bases*, such that:

(B1) $\mathcal{B} \neq \emptyset$.

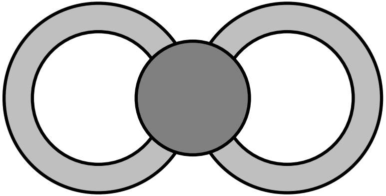
(B2') For all $B_1, B_2 \in \mathcal{B}$ we have $|B_1| = |B_2|$.

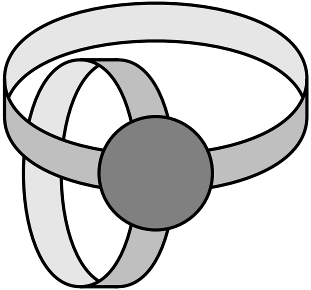
(B3) For all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 - B_2$ there is a $y \in B_2 - B_1$ such that $B_1 - \{x\} \cup \{y\} \in \mathcal{B}$.

Ribbon graphs

Start with a graph and let vertex $\rightarrow$ disk, edge $\rightarrow$ ribbon.

Quasi-spanning tree: subgraph with one boundary component.





Δ-matroid

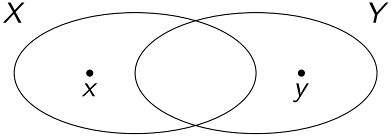
Δ-matroid: a pair $(E, \mathcal{F})$ with

- E finite set;
- $\mathcal{F}$ a family of subsets of E, the *feasible sets*, such that:

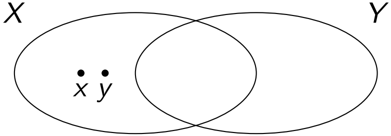
  For all $X, Y \in \mathcal{F}$ and for all $x \in X \triangle Y$ there is a $y \in X \triangle Y$ such that $X \triangle \{x, y\} \in \mathcal{F}$.

Exchange property for feasible sets

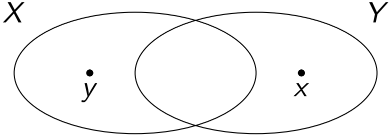
For all $X, Y \in \mathcal{F}$ and for all $x \in X \triangle Y$ there is a $y \in X \triangle Y$ such that $X \cup \{x, y\} \in \mathcal{F}$.



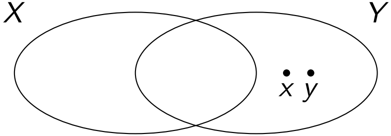
remove x, add y



remove x and y



add x, remove y



add x and y

GOAL: Define the q-analogue of a Δ-matroid.

Problem: we do not have (yet) a q-analogue for graphs, ribbon graphs, nor the symmetric difference.

q-Matroid

q-Matroid: a pair $(E, \mathcal{B})$ with

- E finite space;
- $\mathcal{B}$ a family of subspaces of E, the *bases*, such that:

(B1) $\mathcal{B} \neq \emptyset$

(B2') For all $B_1, B_2 \in \mathcal{B}$ we have $\dim B_1 = \dim B_2$.

(nB3) For all $B_1, B_2 \in \mathcal{B}$, and for each subspace A that has codimension 1 in B_1 there exists $X \subseteq E$ of codimension 1 in E such that $X \supseteq A$, $X \not\supseteq B_2$ and $A + x \in \mathcal{B}$ for all 1-dimensional $x \subseteq E$, $x \not\subseteq X$.

q-Δ-matroid (Ceria, Johnsen, J. (2024))

q-Δ-matroid: a pair $(E, \mathcal{F})$ with

- E finite space;
- $\mathcal{F}$ a family of subspaces of E, the *feasible spaces*, such that:

(F1) For every two subspaces X and Y in $\mathcal{F}$, and for each subspace $A \subseteq E$ that has codimension 1 in X, there either exists:

(i) a codimension 1 space $Z \subseteq E$ with $A \subseteq Z$ and $Y \not\subseteq Z$, such that for all $z \subseteq E$, $z \not\subseteq Z$ it holds that $A + z \in \mathcal{F}$; or

(ii) a codimension 1 space $Z \subseteq E$ such that $Z \cap A \in \mathcal{F}$.

(F2) For every two subspaces X and Y in $\mathcal{F}$, and for each subspace $A \subseteq E$ with X of codimension 1 in A, there either exists:

(iii) a 1-dimensional $z \subseteq E$ with $z \subseteq A$, $z \not\subseteq Y$, such that for each $Z \subseteq E$ of codimension 1, $z \not\subseteq Z$ it holds that $A \cap Z \in \mathcal{F}$; or

(iv) a 1-dimensional $z \subseteq E$ such that $A + z \in \mathcal{F}$.

q-Δ-matroid

Example

Let $E = \mathbb{F}^4$ and $\mathcal{S}$ a *spread* of 2-spaces in E. Let $\mathcal{F} = \mathcal{S} \cup \{0, E\}$.
Then $(E, \mathcal{F})$ is a q-Δ-matroid.

Nice properties of q-Δ-matroids

Theorem (Duality)

Let $(E, \mathcal{F})$ be a q-Δ-matroid and let $\mathcal{F}^{\perp} = \{F^{\perp} : F \in \mathcal{F}\}$. Then $(E, \mathcal{F}^{\perp})$ is a q-Δ-matroid.

Theorem (q-Matroids from q-Δ-matroids)

Let $(E, \mathcal{F})$ be a q-Δ-matroid. Then all feasible spaces of maximal dimension, and all feasible spaces of minimum dimension, are the families of basis of q-matroids.

Theorem (q-Δ-Matroids from q-matroids)

The families of basis, independent spaces, and spanning spaces of a q-matroid all form the family of feasible spaces of a q-Δ-matroid.

Nice properties of q-Δ-matroids

Definition

Let $\varphi : M_1 \to M_2$ be a strong map between q-matroids. Then the family of all spaces contained in a basis of M_1 and containing a basis of M_2, are the feasible spaces of a *q-g-matroid*.

Theorem

Every q-g-matroid is a q-Δ-matroid.

To do list / open questions

- Restriction and contraction of q-Δ-matroids
- Partial duality for q-Δ-matroids
- Is there always a strong map between the upper and lower q-matroid of a q-Δ-matroid?
- q-Analogue of birank for Δ-matroids
- Dream: q-analogue of graph embedding / ribbon graphs